M.Sc

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MATHEMATICAL PHYSICS

MATHEMATICS

Original Notes

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INITIAL VALUE & BOUNDARY VALUE PROBLEMS

ASSOCIATED WITH HEAT & WAVE EQS.

Ordinary differential Equations.

An equation involving ordinary derivatives or differentials i.e. derivative w.r.t. one variable is called an ordinary differential equation.

e.g.

$$\frac{dy}{dx} + 7y = 0$$

$$x\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 3xy = 17$$

are ordinary differential equations.

Partial differential Equation.

An eq. involving partial derivatives i.e. derivatives w.r.t. more than one variables is called Partial diff. equation. e.g.

$$u_{xx} = \alpha^2 u_t$$

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

are partial differential Equations

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Order of Diff. Eq.

The highest derivative involved in the diff. eq. is called order of the diff. eq. e.g.

$$\frac{dy}{dx} - 3y = 0$$

& $$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + 8y = 0$$

are diff. eqs. of order 1 and 2 respectively.

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Degree of Diff. Eq.

The highest exponent of the highest derivative involved in the diff. equation is called degree of the diff. eq. e.g.

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 - 8y = 0$$

and $\frac{dy}{dx} + 3xy = 0$ are diff. eqs. of orders 2 and 1 respectively.

General Solution.

A linear combination of all possible solutions of a diff. equation is called general solution of the eq.

Exp. Suppose $\frac{d^2y}{dx^2} - 4y = 0$
Then it can easily be seen that $y = e^{2x}$ & $y = e^{-2x}$ are two its solutions. So its general solution is given by

$$y = Ae^{2x} + Be^{-2x}$$

Linear Differential Equation.

A diff. eq is said to be linear if

(1) The dependent variable y and all of its derivatives are of degree 1.

(2) No product of y and/or any of its derivative occures.

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(3) No transcedental function of y and/or any of its derivative occures.

A differential equation which is not linear is said to be non-linear.

Examples:-

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1:- $U_{xx}+U_{yy}+U_{zz}=0$

2:- $U_{xx}+U_{yy}=\alpha^2 U_t$

3:- $U_{xx}=\frac{1}{c^2}U_t$

are all linear differential eqs. of order 2 and degree 1.

4:- $uU_{xx}+zU_{xy}=0$

5:- $U_{xx}+U_{yy}+u^2=0$

6:- $U_{xx}+U_{xy}+\tan u=0$

are all non-linear diff. eqs of order 2 and degree 1

Initial Value and Boundary Value Problems

A diff. eq. with conditions at one initial point e.g.

$u(x_0)=u_0$, $u'(x_0)=u_0$

is called initial value problem.

A diff. eq. with conditions at two different points e.g.

$u(x_0)=u_0$, $u(x_1)=u_1$

or $u'(x_0)=u_0$ $u'(x_1)=u_1$

is called boundary value problem.

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Operator:-

If A & B are two classes of functions. A rule which assigns each function of A to a unique function of B is called operator.

Examples

1:- The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplace operator.

2:- The operator $\nabla^2 - \alpha^2 \partial/\partial t$ is called heat operator.

3:- The operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called wave operator.

Linear Operator.

An operator A is said to be linear if

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$$

Otherwise it is said to be non linear.

Exps.

1:- Laplace operator is linear.

2:- operator $\left(\frac{d}{dx}\right)^2$ is non linear.

Sum of Linear Operators.

If A & B are two linear operators then their sum is defined as

$$(A \pm B)(f) = A(f) \pm B(f).$$

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Definition. If L is a linear operator and f is a function, then the equation $L(u) = f$ is called a linear homogeneous eq., otherwise it is said to be non homogeneous.

Principal of Superposition.

Let $f_1, f_2, \dots, f_n$ be any functions and $c_1, c_2, \dots, c_n$ be any constants.

Let $u_1, u_2, \dots, u_n$ be the functions s.t. $L(u_i) = f_i \quad 1 \leq i \leq n.$

Then $L(u) = f$

Where $u = c_1u_1 + c_2u_2 + \dots + c_nu_n$

$f = c_1f_1 + c_2f_2 + \dots + c_nf_n$

Proof:-

$$
\begin{aligned}
L(u) &= L(c_1u_1 + c_2u_2 + \dots + c_nu_n) \\
&= L(c_1u_1) + L(c_2u_2) + \dots + L(c_nu_n) \\
&= c_1L(u_1) + c_2L(u_2) + \dots + c_nL(u_n) \\
&= c_1f_1 + c_2f_2 + \dots + c_nf_n \\
&= f
\end{aligned}
$$

$\Rightarrow L(u) = f$

Derivation of Laplace Eq.

According to Gruass's flux Theorem in electrostatic

"Total flux of electrostatic intensity $\bar{E}$ across a closed surface S is equal to 4π times the total charge enclosed within the surface S."

Mathematically

$$\iint_S E \cdot ds = \iiint_V 4\pi \rho \, dv \longrightarrow (1)$$

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Where ρ is volume density
But by Guass's Divergence Theorem

$$\iint_S \bar{E}\cdot ds = \iiint_V \text{div}\, \bar{E}\, dv \longrightarrow (2)$$

From (1) & (2)

$$\iiint_V \text{div}\, \bar{E}\, dv = \iiint_V 4\pi\rho\, dv$$

$$\Rightarrow \iiint (\text{div}\, \bar{E} - 4\pi\rho) dv = 0$$

$$\Rightarrow \text{div}\, \bar{E} - 4\pi\rho = 0 \qquad \because dv \neq 0$$

$$\Rightarrow \text{div}\, \bar{E} = 4\pi\rho$$

$$\Rightarrow \nabla\cdot\bar{E} = 4\pi\rho$$

But by Fouriers Law

$$\bar{E} = -\nabla\varphi \quad \text{So}$$

$$\nabla\cdot(-\nabla\varphi) = 4\pi\rho$$

$$\Rightarrow \nabla^2\varphi = -4\pi\rho \quad \text{(Poisson Eq.)}$$

Now for the region where there is no charge, $\rho = 0$, then

$$\nabla^2\varphi = 0$$

This eq. is called Laplace eq.

DERIVATION OF HEAT Equation.

The Fourier law states that

$$E = -k\nabla T$$

Where E is intensity i.e. flow of heat, T is temperature and k is the constant of conductivity. The $-ve$ sign shows the direction of flow of heat.

Consider a closed surface enclosing a volume V, then the total outflow of heat energy across the surface S per unit time is equal to

$$-\iint_S \bar{E}\cdot ds = -\iiint_V \text{div}\,\bar{E}\, dv$$

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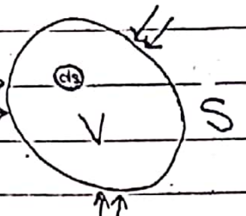
Here we have applied Guass's Divergence Theorem



$$\Rightarrow -\iint_S \bar{E}\cdot ds = -\iiint_V \operatorname{div}(-k\nabla T)\,dv$$

$$\because \bar{E} = -k\nabla T$$

$$\Rightarrow -\iint_S \bar{E}\cdot ds = k\iiint_V \nabla^2 T\,dv$$

Now according to Law of conservation of energy.

Outflow of heat = Inflow of heat

If C_p is the specific heat of the body (i.e. C_p is the heat required to raise the temperature of unit mass of a body through 1°C), then the total heat energy required to raise the temperature of an element of mass $\rho\,dv$ to the temperature T°C is

$\rho\,dv\,C_p T$. Where ρ is density of element.

Thus

rate of increase of temperature per unit time is $\frac{\partial}{\partial t}\iiint_V C_p T \rho\,dv$

Which is also total inflow of heat.

Now by ①

$$k\iiint_V \nabla^2 T\,dv = \frac{\partial}{\partial t}\iiint_V C_p T \rho\,dv$$

$$\Rightarrow \iiint_V \left(k\nabla^2 T - \rho C_p \frac{\partial T}{\partial t}\right) dv = 0$$

$$\Rightarrow k\nabla^2 T - \rho C_p \frac{\partial T}{\partial t} = 0 \qquad \because dv \neq 0$$

$$\Rightarrow \nabla^2 T = \alpha^2 T_t \qquad \text{where } \alpha^2 = \frac{\rho C_p}{k}$$

Which is required heat conduction equation.

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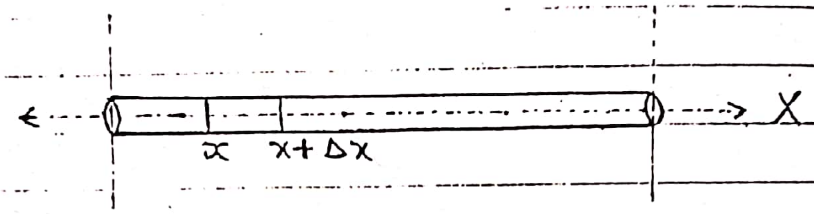
Q. Derive heat equation for the flow of heat through a rod of uniform thickness taking into consideration source of heat.

SOLUTION.

Consider one dimensional flow of heat in a rod of uniform cross sectional area. Let A be the area



of cross section of the rod and let x-axis be chosen along the length of the rod.

We use law of conservation of energy to a small portion of rod. This law may be stated as

" Amount of heat energy that enters a region + amount of energy generated inside the region in a given time = Amount of energy that leaves the region + the amount of energy absorbed or stored." → (1)

Let $q(x,t)$ be the flow of heat at point x at time 't' then the rate at which heat is stored in small volume element

= specific heat × mass of material × rise in temperature in one sec

$$= C_p \times A\,\Delta x\,\rho \times \frac{\partial T}{\partial t}$$

$(\because \text{mass} = \text{Vol.} \times \text{Density} = A\Delta x \times \rho)$

Now the rate at which heat enters the volume element $= A\,q(x,t)$
& rate at which heat leaves the volume element $= A\,q(x+\Delta x,t)$

Now let r be the rate of generation of heat energy per unit volume, then
heat generation inside volume element $= r A \Delta x$

Putting these values in ①

$$A\,q(x,t) + r A \Delta x = A\,q(x+\Delta x,t) + A\Delta x\, C_p \rho \frac{\partial T}{\partial t}$$

÷ing both sides by A

$$q(x,t) + r\,\Delta x = q(x+\Delta x,t) + \Delta x\, C_p \rho \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} = r - C_p \rho \frac{\partial T}{\partial t}$$

Taking limit as $\Delta x \to 0$

$$\frac{\partial q}{\partial x} = r - C_p \rho \frac{\partial T}{\partial t}$$

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$$\Rightarrow \frac{\partial}{\partial x}\left(-k \frac{\partial T}{\partial x}\right) = r - C_p \rho \frac{\partial T}{\partial t}$$

$\because$ by fourier law $q = -k \frac{\partial T}{\partial x}$

$$\Rightarrow -k \frac{\partial^2 T}{\partial x^2} = r - C_p \rho \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} - \frac{\rho C_p}{k} \frac{\partial T}{\partial t} = -\frac{r}{k} \qquad \div\text{ing by } -k$$

Which is one dimensional form of heat equation in the presence of source of heat. If there is no source of heat then $r = 0$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{\rho C_p}{k} \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}$$

Where $\alpha^2 = \frac{k}{\rho C_p}$

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Solution of Heat Eq.

The conduction of heat is described by the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \rightarrow (1)$$

with initial condition

$$u(x,0) = f(x) \rightarrow (2)$$

& B.Cs. $u(0,t) = T_0 \rightarrow (3)$ & $u(a,t) = T_1 \rightarrow (4)$

Here B.Cs. are non homogenous

If $T_0 = 0$, $T_1 = 0$

Then the B.Cs. are said to be homogeneous.

To solve the partial diff. equation by using the standard method of seperation of variables. 1st the B.Cs. are made homogeneous if they are not homogeneous.

STEADY STATE OR EQUILIBRIUM STATE

مستقل Distribution.

If u depends upon t also then the flow of heat is time dependent or non steady. And when $t \to \infty$ i.e. after a long time the non steady temperature tends to become steady. If $V(x)$ represents steady flow of temperature then

$$u(x,t) \rightarrow V(x) \quad \text{when } t \to \infty$$

Now if we put

$\lim_{t\to\infty} u(x,t) = V(x)$, then

from ①

$$\frac{d^2V}{dx^2} = 0 \rightarrow ⑤ \quad \because \frac{\partial u}{\partial t} = 0 \text{ as } t \to \infty$$

From ③ $V(0) = T_0 \rightarrow ⑥$

From ④ $V(a) = T_1 \rightarrow ⑦$

Now from ⑤ $V(x) = Ax + B$

$V(0) = T_0 \Rightarrow B = T_0$

$V(a) = T_1 \Rightarrow Aa + B = T_1 \Rightarrow A = \frac{T_1 - T_0}{a}$

Hence

$$V(x) = \left(\frac{T_1 - T_0}{a}\right)x + T_0 \rightarrow ⑧$$

Which is req. Solution of steady state problem.

TRANSIENT TEPRATURE DISTRIBUTION

If we define $w(x,t) = u(x,t) - V(x)$

Then $\lim_{t\to\infty} w(x,t) = V(x) - V(x) \rightarrow ⑨$

$= 0$

i.e. $w(x,t)$ is non zero only for that values of t which are not very large. $w(x,t)$ is called transient temperature distribution.

Now from ⑨

$$u(x,t) = w(x,t) + V(x)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial V}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} + 0 \quad \because \frac{d^2V}{dx^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}$$

Also $\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}$

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Using these values eq. ① takes the form

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} \rightarrow (10)$$

Now by ⑨

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) = g(x)$$

$$\Rightarrow w(x,0) = g(x)$$

$$w(0,t) = u(0,t) - v(0) = T_0 - T_0 = 0$$

$$w(a,t) = u(a,t) - v(a) = T_1 - T_1 = 0$$

Hence by ⑦ the given problem reduces to

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t}$$

$$w(x,0) = f(x) - v(x).$$

$$w(0,t) = 0 \qquad w(a,t) = 0$$

Now the B.Cs. are homogenous, so the system can be solved by standard method of seperation of variables. (S.O.V.)

Q. Solve by using method of seperation of variables.

$$u_{xx} = \alpha^2 u_t \rightarrow ①$$

Sol. Let $u = X(x)T(t)$

$$\Rightarrow u_{xx} = X_{xx}T, \quad u_t = XT_t$$

Hence from ①

$$X_{xx}T = \alpha^2 X T_t$$

$$\Rightarrow \frac{X_{xx}}{X} = \alpha^2 \frac{T_t}{T} = \lambda^2$$

$$\Rightarrow \frac{X_{xx}}{X} = \lambda^2 \Rightarrow X_{xx} - \lambda^2 X = 0$$

$$\Rightarrow (D^2 - \lambda^2)X = 0$$

A. Eq. is $D^2 - \lambda^2 = 0$

$$\Rightarrow D = \pm\lambda$$

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So $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

Now $\alpha^2 \frac{T_t}{T} = \lambda^2 \Rightarrow T_t - \frac{\lambda^2}{\alpha^2} T = 0$

$\Rightarrow \int \frac{T_t}{T} dt = \int \frac{\lambda^2}{\alpha^2} dt + \ln C$

$\Rightarrow \ln T = \frac{\lambda^2}{\alpha^2} t + \ln C$

$\Rightarrow \ln \frac{T}{C} = \frac{\lambda^2}{\alpha^2} t$

$\Rightarrow T = C e^{(\lambda^2/\alpha^2)t}$

Hence

$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{(\lambda^2/\alpha^2)t})$

Which is req. sol.

V. Imp.

Q. Discuss the Dirichlet, Neumann, Robin and Mixed boundary conditions associated with heat equation as well as the initial condition.

(91, 93, 94, 95, 96)

Sol.

There must be one initial and two boundary conditions to solve the problem uniquely. Such conditions are usually of the form $u(x,0) = F(x)$, which gives the initial temperature distribution. Now the associated boundary conditions can be divided into four different types.

1:- Dirichlet's condition or B.Cs. of 1st kind.

These conditions can be described by $u(0,t) = T_0$ & $u(a,t) = T_1$

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$t > 0$, where T_0 & T_1 are the temperatures and $x = 0, x = a$ are the end points of rod. Here T_0 & T_1 may be different or same.

Neumann's Conditions or B.Cs. of 2nd kind.

Another possibility is that that the rate of flow of heat is specified at one or more boundary points. Since rate of flow is related to gradient of temperature by Fourier's law (i.e. $E = -k\nabla T$) such a condition can be written as

$$u_x(0,t) = \gamma(t) \quad \& \quad u_x(a,t) = \delta(t)$$

Where in general γ & δ ~~may be constants, or zero~~ are functions of 't' and in particular γ & δ may be constants or zero. If $\gamma = 0$ then there is no flow at $x = 0$

Robin's Conditions or B.Cs. of 3rd kind.

The boundary conditions of the form

$$a_1 u(0,t) + a_2 u_x(0,t) = \text{constant}$$
$$b_1 u(a,t) + b_2 u_x(a,t) = \text{constant}$$

are called Robin's conditions or B.Cs. of 3rd kind.

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فزیکل مینِنگز آف رابنز کنڈیشنز:

According to the Newton's Law of cooling, "If a hot body is in contact with a less hot body (sourrounding body), it loses heat by convention. Rate of transfer of heat is proportional to the difference of temperature b/w the two bodies i.e.

$$q(x,t) \propto u - T$$

$$\Rightarrow q(x,t) = h(u - T)$$

Applying this law at end point $x = x_0$

Then $q(x_0,t) = h[u(x_0,t) - T]$

$$\Rightarrow -k\frac{\partial u}{\partial x}(x_0,t) = h\{u(x_0,t) - T\}$$

$\because$ by fourier's Law $q = -k\frac{\partial u}{\partial x}$

$$\Rightarrow hu(x_0,t) + k\frac{\partial u(x_0,t)}{\partial x} = hT$$

Which is Robin's condition at $x = x_0$.

MIXED BOUNDARY CONDITIONS:-

All the above boundary conditions involve the function u and its derivatives at one point. If more than one point is involved then the B.C. is called Mixed B.C.

e.g. if a uniform rod is bent s.t. the ends $x=0$ and $x=a$ jointed together. Then appropriate mixed B.Cs. are

$$u(0,t) = u(a,t) \quad , \quad \frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(a,t) \quad ; \quad t \geq 0$$

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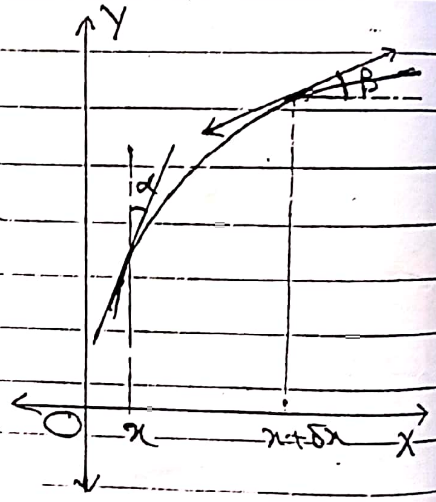
DERIVATION OF WAVE Eq.

We shall derive the wave equation by applying the law of motion to a small element of the string b/w x and $x+\delta x$. Also $u(x,t)$ will denote the displacement at the point x and at time t.

Let $T(x)$ denotes the tension in the string at time 't' and at a point x. And $T(x+\delta x)$ denotes the tension at other end at the same time.



Now as weight is acting downward and no external force is suppose to be present. Then the total force in the x-direction must be zero.

$$\Rightarrow -T(x)\cos\alpha + T(x+\delta x)\cos\beta = 0$$

$$\Rightarrow T(x)\cos\alpha = T(x+\delta x)\cos\beta = T \text{ (say)}$$

$$\Rightarrow T(x) = T/\cos\alpha$$

$$\& \ T(x+\delta x) = T/\cos\beta$$

Similarly considering the force on the string in the vertical direction.

$$-T(x)\sin\alpha + T(x+\delta x)\sin\beta - mg = m\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow -\frac{T}{\cos\alpha}\sin\alpha + \frac{T}{\cos\beta}\sin\beta - mg = m\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow -T\tan\alpha + T\tan\beta - mg = m\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

But

$\tan\alpha$ = Slope of string at $x = \frac{\partial u(x,t)}{\partial x}$

$\tan\beta$ = slope of string at $x+\delta x = \frac{\partial u(x+\delta x,t)}{\partial x}$

So by ①

$-T\,u_x(x,t) + T\,u_x(x+\delta x,t) - mg = m\,u_{tt}$

Now as $m = \rho\,\delta x$

$\Rightarrow -T\,u_x(x,t) + T\,u_x(x+\delta x,t) - \rho\,\delta x\,g = \rho\,\delta x\,u_{tt}$

$\Rightarrow \frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x} = \frac{\rho g}{T} + \frac{\rho}{T}u_{tt}$

Applying limit as $\delta x \to 0$

$u_{xx}(x,t) = \frac{\rho g}{T} + \frac{\rho}{T}u_{tt}$

Now $T/\rho = c^2$

$\Rightarrow u_{xx} = \frac{g}{c^2} + \frac{1}{c^2}u_{tt}$

$\Rightarrow u_{xx} = \frac{1}{c^2}u_{tt}$ $\quad \because g <<< \cdots < c^2$

Which is required wave eq. in one dimension.

MODIFIED FORM OF WAVE EQ.

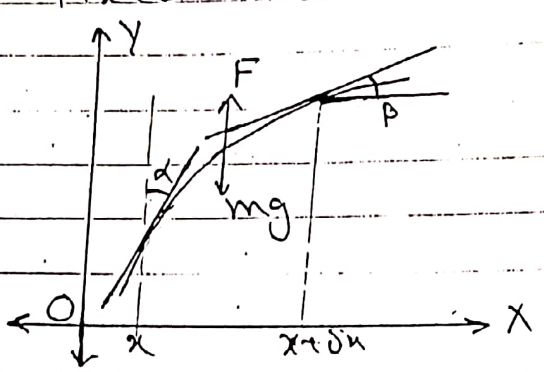
Suppose a distributed vertical force $F(x,t)$ (+ve upward) acts on the string. Then the eq. of motion is

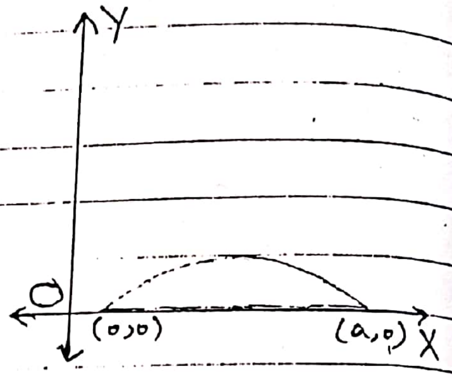
$$u_{xx} = \frac{1}{c^2}u_{tt} - \frac{F(x,t)}{T}$$

Derivation.

Let us consider the vibration of a stretched string of length a. Suppose that the string is completely flexible and has uniform density ρ, and is

fixed at $x=0$ and at $x=a$





We shall derive the equation by applying the law of motion to a small element of the string lying b/w x and $x+\delta x$. Also $u(x,t)$ denotes the displacement at the point x and at time t.

Let $T(x)$ and $T(x+\delta x)$ be the tensions in string at the points x and $x+\delta x$

Then for the forces acting in horizontal direction:

$$-T(x)\cos\alpha + T(x+\delta x)\cos\beta = 0$$

$$\Rightarrow T(x)\cos\alpha = T(x+\delta x)\cos\beta = T \text{ (say)}$$

$$\Rightarrow T(x) = T/\cos\alpha, \quad T(x+\delta x) = T/\cos\beta \} \rightarrow (1)$$

For vertical forces

$$-T(x)\sin\alpha + T(x+\delta x)\sin\beta - mg + F\delta x = m\,u_{tt}$$

Using (1)

$$-T\tan\alpha + T\tan\beta - mg + F\delta x = m\,u_{tt}$$

$$\Rightarrow -T\,u_x(x,t) + T\,u_x(x+\delta x,t) - \rho\,\delta x\,g + F(x,t)\,\delta x = \rho\,\delta x\,u_{tt} \} \rightarrow (2)$$

$\because$ $\tan\alpha$ = slope of string at $x = u_x(x,t)$

$\tan\beta$ = " " " at $x+\delta x = u_x(x+\delta x,t)$

& $m = \rho\,\delta x$

$$\Rightarrow \lim_{\delta x \to 0} \frac{U_x(x+\delta x,t) - U_x(x,t)}{\delta x} = \frac{\rho g}{T} - \frac{F}{T} + \frac{\rho}{T} U_{tt}$$

$$\Rightarrow U_{xx} = \frac{1}{c^2} U_{tt} - \frac{F(x,t)}{T}$$

Which is req. wave equation in the presence of external force $F(x,t)$. In the absence of external force

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

Remark: If the distributed vertical force is the load $W(x,t)$, acting downward. Then replacing $F(x,t)$ by $-W(x,t)$ we get the eq.

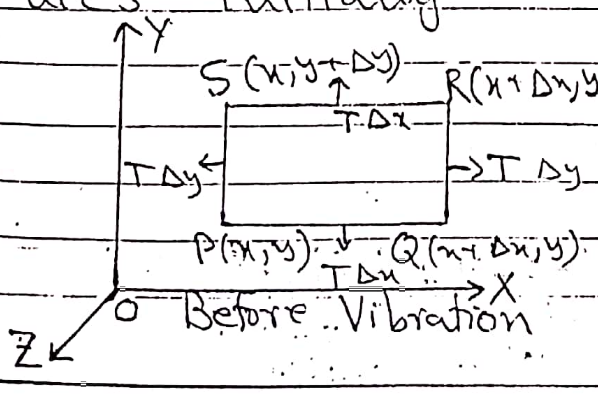
$$U_{xx} = \frac{1}{c^2} U_{tt} + \frac{W(x,t)}{T}$$

TWO DIMENSIONAL WAVE Equation

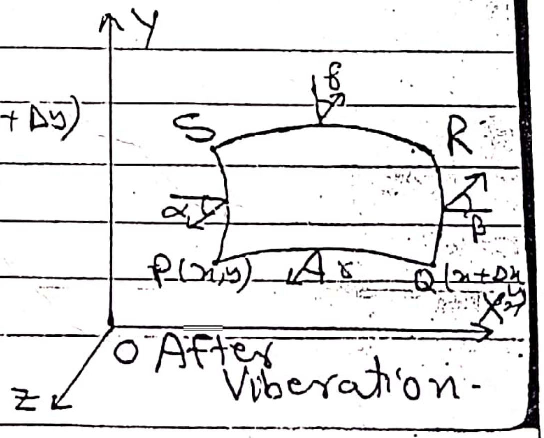
91, 93, 94

To derive the two dimensional wave equation we consider the viberation of a stretched membrane which is completely flexible, has negligible mass and has uniform density ρ.

Further suppose that the displacements of the membrane are small and are always $\perp$ to the plane in which it lies initially.



Before Vibration



After Viberation

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Let the membrane initially lies in XY-plane and T denotes the surface tension. To derive the equation we consider a small element PQRS of membrane with sides Δx and Δy. Let the forces act along tangent planes to the sides.

Let α, β be the angles b/w XY-plane and tangent planes at PS and QR respectively and let γ, δ be the angles b/w XY-plane and tangent planes at PQ and SR resp.

Now for the forces acting along z-axis

$$\Delta y\, T\cdot(\sin\beta - \sin\alpha) + \Delta x\, T\cdot(\sin\gamma - \sin\delta) = \Delta x \Delta y\, \rho \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\Delta x} + \frac{\sin\gamma - \sin\delta}{\Delta y} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \rightarrow (1)$$

But as α, β, γ & δ are very small.

So $\sin\alpha \simeq \tan\alpha = \frac{\partial u}{\partial x}(x,y,t)$ etc.

$\Rightarrow$

$$\frac{u_x(x+\Delta x, y, t) - u_x(x,y,t)}{\Delta x} + \frac{u_y(x, y+\Delta y, t) - u_y}{\Delta y} = \frac{\rho}{T} u_{tt}$$

$\Rightarrow u_{xx} + u_{yy} = \frac{\rho}{T} u_{tt}$ As $\Delta x, \Delta y \rightarrow 0$

$\Rightarrow u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$ $\because \frac{1}{c^2} = \frac{\rho}{T}$

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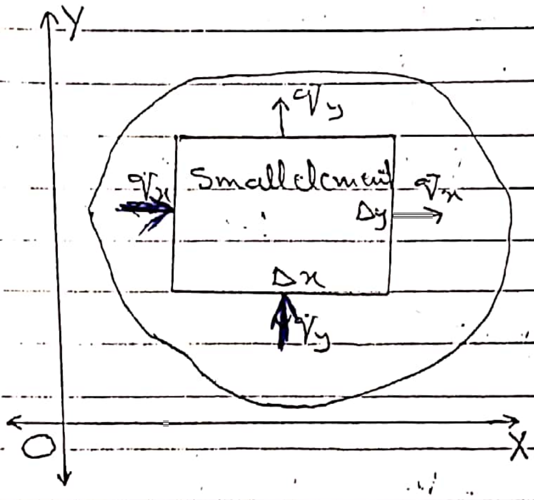
2-DIMENSIONAL HEAT Equation

Consider a plate in XY-plane as shown in figure.



In order to derive the eq. we apply the law of conservation of energy to small element of plate in the form of rectangle with dimensions Δx and Δy. According to this law

" Total energy that enters the body + Total energy that generates inside the body = Total energy that leaves the body + Total energy stored in body "

Let $q_x(x,y,t)$ and $q_y(x,y,t)$ denotes the flow of heat in the directions of x-axis and y-axis respectively.

Let θ be the thickness of the plate, r be rate of generation per unit volume, ρ is density and c be heat capacity, then

rate of generation $= r\theta\,\Delta x\,\Delta y$

" " Storage $= \rho c\theta\,\Delta x\,\Delta y\,u_t$

Rate at which heat enters body

$$= q_x\,\theta\,\Delta y + q_y\,\theta\,\Delta x$$

Rate at which heat leaves the body

$$= q_x(x+\Delta x,y,t)\,\Delta y\,\theta + q_y(x,y+\Delta y,t)\cdot\theta\,\Delta x$$

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So by law of conservation of energy

$q_x(x,y,t)\,\theta\,\Delta y + q_y(x,y,t)\,\theta\,\Delta x + r\,\theta\,\Delta x\,\Delta y$
$= q_x(x+\Delta x,y,t)\,\theta\,\Delta y + q_y(x,y+\Delta y,t)\,\theta\,\Delta x$
$+ \rho\,\theta\,c\,\Delta x\,\Delta y\,u_t$

$$\Rightarrow \frac{q_x(x+\Delta x,y,t)-q_x(x,y,t)}{\Delta x} + \frac{q_y(x,y+\Delta y,t)-q_y(x,y,t)}{\Delta y} = r - \rho c u_t$$

Applying limits as $\Delta x, \Delta y \to 0$

$$\Rightarrow \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = r - \rho c \frac{\partial u}{\partial t} \to (1)$$

By Fourier's Law

$q_x = -k_x \frac{\partial u}{\partial x}$ $\quad q_y = -k_y \frac{\partial u}{\partial y}$

Then by (1) $-k_x u_{xx} - k_y u_{yy} = r - \rho c u_t$

$\Rightarrow k_x u_{xx} + k_y u_{yy} = \rho c u_t - r$

In general k_x & k_y are different.

If $k_x = k_y = k$ (say)

Then

$$u_{xx} + u_{yy} = \frac{\rho c}{k} u_t - \frac{r}{k}$$

In absence of source of heat

$u_{xx} + u_{yy} = \frac{1}{\alpha^2} u_t$ $\quad \because \alpha^2 = k/\rho c$ & $r = 0$

As req.

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Q:- Find the steady Solution of the eq.

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \quad t>0$$

subject to $u_x(0,t)=0$ $\quad u_x(a,t)=0$

$u(x,0) = f(x)$ $\quad 0<x<a$

Also discuss the uniqueness of the solution.

Solution.

When $t\to\infty$ then non steady sol. tends to the steady solution.

$\therefore$ put $\lim_{t\to\infty} u(x,t) = V(x)$

Where $V(x)$ is the steady solution.

Using this $u_{xx} = \frac{1}{k} u_t$ becomes

$$\frac{d^2V}{dx^2} = 0$$

$$\Rightarrow V(x) = Ax+B \quad , V_x(x) = A$$

Also $u_x(0,t)=0 \Rightarrow V_x(0)=0 \Rightarrow A=0$

$u_x(a,t) = 0 \Rightarrow V_x(a)=0 \Rightarrow A=0$

So $V(x)=B$

Since for different values of B $V(x)$ is different and so is not unique. However by physical consideration we can determine the constant (i.e. its unique value) and therefore the unique solution.

The problem describes the flow of heat in a rod of length 'a' which is insulated at the sides at the end points $x=0$, $x=a$. It means that the heat contained in the rod is constant for all the time e.g. at $t=0$ and at $t=\infty$

Now heat stored in time

t is $\int_0^a \rho C A\, u(x,t)\, dx = \rho C A \int_0^a u(x,t)\, dx$

This quantity is same at $t=0, t=\infty$

$\rho AC \int_0^a u(x,0)\,dx = \rho C A \lim_{t\to\infty} \int_0^a u(x,t)\,dx$

$\Rightarrow \int_0^a f(x)\,dx = \int_0^a V(x)\,dx = \int_0^a B\,dx = Ba$

$\Rightarrow B = \frac{1}{a}\int_0^a f(x)\,dx$

So unique sol is $V(x) = \frac{1}{a}\int_0^a f(x)\,dx$

Q. Find steady state sol. of the problem $u_{xx} = \frac{1}{k} u_t \rightarrow (1)$, $0<x<a, t\geq 0$

$u(0,t) = T_0 \rightarrow (2)$

$-k\, u_x(a,t) = h[u(a,t) - T_1] \rightarrow (3)$

$u(x,0) = f(x) \rightarrow (4)$ $0<x<a$

Sol. Put $\lim_{t\to\infty} u(x,t) = V(x)$

Where $V(x)$ is the steady state sol.

Now by (1) $\frac{d^2V}{dx^2} = 0$

$\Longrightarrow V(x) = Ax + B$

$u(0,t) = T_0 \Rightarrow V(0) = T_0 \Rightarrow B = T_0$

$-k\, u_x(a,t) = h[u(a,t) - T_1]$

$-kA = h[Aa + T_0 - T_1]$

$\Rightarrow A = \frac{T_1 - T_0}{ha + k}$

So required steady state solution is

$V(x) = \left(\frac{T_1 - T_0}{ha+k}\right)x + T_0$

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Q. Interpret and solve the problem
$u_{xx} = \frac{1}{k} u_t$ $0<x<a,\ t>0$
$u(0,t) = T_0$, $u(a,t) = T_1$ $u(x,0) = f'(x)$

Solution:

PHYSICAL INTEPRETATION

Obviously given problem describes the flow of heat in a rod of length 'a' which is included along the sides and whose end points are maintained at temperatures T_0 and T_1. The original temperature of the distribution is $f'(x)$.

1st we obtain steady state sol. by using $\lim_{t\to\infty} u(x,t) = V(x)$

Then given eq. becomes:

$$\frac{d^2V}{dx^2} = 0 \Rightarrow V(x) = Ax + B$$

$$V(0) = T_0 \Rightarrow B = T_0$$

$$V(a) = T_1 \Rightarrow A = \frac{T_1 - T_0}{a} \Rightarrow V(x) = \left(\frac{T_1 - T_0}{a}\right)x + T_0$$

Now we obtain transient temp. distribution by substitution

$$u(x,t) = w(x,t) + V(x)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial V}{\partial x} \quad \& \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} + 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} \quad \& \text{ also } \frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}$$

Hence we have

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{K}\frac{\partial w}{\partial t}$$

Then $w(x,t) = u(x,t) - V(x)$

$w(0,t) = u(0,t) - V(0) = T_0 - T_0 = 0$

$w(a,t) = u(a,t) - V(a) = T_1 - T_0 + T_0 - T_1 = 0$

$w(x,0) = u(x,0) - V(x) = f'(x) - V(x) = g(x)$

Hence given pb. becomes $w_{xx} = \frac{1}{k} w_t$ with $w(0,t) = 0$, $w(a,t) = 0$; $w(x,0) = g(x)$ and can be solved by method of S.O.V.

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Q. Find the steady state soln. of the problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = c\rho\frac{\partial u}{\partial t} \qquad 0<x<a, \; t>0$$

$u(0,t) = T_0$, $u(a,t) = T_1$
If $k(x) = b + dx$, b, d are constants.

Sol.
Put $\lim_{t\to\infty} u(x,t) = V(x)$
Then $\frac{d}{dx}\left(k\frac{dV}{dx}\right) = 0 \rightarrow ①$

$\Rightarrow k\frac{dV}{dx} = A \quad \Rightarrow \frac{dV}{dx} = \frac{A}{k} = \frac{A}{b+dx}$

$\Rightarrow V = \frac{A}{d}\ln|b+dx| + B \rightarrow *$

Now
$u(0,t) = T_0 \Rightarrow V(0) = T_0$
$\Rightarrow \frac{A}{d}\ln|b| + B = T_0$
$\Rightarrow B = T_0 - \frac{A}{d}\ln|b| \rightarrow ②$

$u(a,t) = T_1 \Rightarrow V(a) = T_1$
$\Rightarrow \frac{A}{d}\ln|b+ad| + B = T_1$
$\Rightarrow \therefore A = \frac{d(T_1 - T_0)}{\ln|1+\frac{ad}{b}|}$ & $B = T_0 - \frac{\ln|b|(T_1-T_0)}{\ln|1+\frac{ad}{b}|}$

Hence by * req. S.S.S. is

$$V(x) = \frac{T_1 - T_0}{\ln|1+\frac{ad}{b}|}\ln|b+dx| + T_0 - \frac{\ln|b|\,(T_1-T_0)}{\ln|1+\frac{ad}{b}|}$$

WRONSKIAN

Wronskian of two functions f & g is denoted and defined by

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

Theorem.
Let the functions $f_1, f_2, f_3, \ldots, f_n$ have derivatives upto '$n-1$' order

and
$$\begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ f_1' & f_2' & f_3' & \cdots & f_{n-1}' & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_{n-1}'' & f_n'' \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} & \cdots & f_{n-1}^{n-1} & f_n^{n-1} \end{vmatrix} \neq 0$$

for at least one value of $x \in [a,b]$ then functions are linearly independent.

Proof. We prove the theorem for $n=2$

Given $W(f_1,f_2) \neq 0$

suppose f_1 & f_2 are linearly dependent

Then $\exists$ c_1 & c_2 (not both zero) s.t.

$c_1f_1 + c_2f_2 = 0 \quad \rightarrow (1)$

& $c_1f_1' + c_2f_2' = 0 \quad \rightarrow (2)$

Since c_1 & c_2 are not zero simultaneously

So $\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$

$\Rightarrow W(f_1,f_2)$

A contradiction. So our supposition is wrong and hence f_1 & f_2 are linearly independent.

Question:- Solve $U_{xx} = \frac{1}{c^2} U_{tt}$ $\rightarrow$ (I)

with $u(0,t) = 0$ $\quad u(l,t)=0 \quad u(x,0) = \phi(x)$

$\dot{u}(x,0) = 0$ $\quad$ (87, 89, 90, 91, 92)

Solution. Since boundary conditions are homogeneous so we use method of S.O.V. So let $U = XT = X(x)\,T(t)$

Then given eq. becomes

$X_{xx}T = \frac{1}{c^2} XT_{tt}$

~~$\Rightarrow X_{xx}T = \frac{1}{c^2} XT_{tt} = \lambda$~~

$\Rightarrow \frac{X_{xx}}{X} = \frac{1}{c^2}\frac{T_{tt}}{T} = \lambda^2$

$\Rightarrow X_{xx} - \lambda^2 X = 0$

$\Rightarrow$ A.E. is $D^2 - \lambda^2 = 0 \quad \Rightarrow D = \pm\lambda$

$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \rightarrow (L)$

Also $\frac{1}{c^2}\frac{T_{tt}}{T} = \lambda^2$

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Traders HOUSE OF QUALITY NOTES

$\Rightarrow (D^2 - c^2\lambda^2)T = 0$

A.E. is $D^2 - c^2\lambda^2 = 0 \Rightarrow D = \pm c\lambda$

So $T(t) = De^{c\lambda t} + Ee^{-c\lambda t}$ → ②

From ① & ②

$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(De^{c\lambda t} + Ee^{-c\lambda t})$ → ③

$u(0,t) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

Hence by ① $X(x) = A(e^{\lambda x} - e^{-\lambda x})$

Also $u(l,t) = 0 \Rightarrow X(l) = 0$?

$\Rightarrow A(e^{\lambda l} - e^{-\lambda l}) = 0 \Rightarrow A = 0$
$\Rightarrow B = 0$

$\Rightarrow u = 0$ which is trivial solution.

For non trivial solution we take

$\frac{X_{xx}}{X} = \frac{1}{c^2}\frac{T_{tt}}{T} = -\lambda^2$

Then $X = A\cos\lambda x + B\sin\lambda x$

$T = D\cos(\lambda c)t + E\sin(c\lambda)t$

Now $u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$

$u(l,t) = 0 \Rightarrow X(l) = 0 \Rightarrow B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0 \quad \because B \neq 0$

$\Rightarrow \lambda = \frac{m\pi}{l} \quad m = \pm 1, \pm 2, \ldots$

So $X_m(x) = B\sin\left(\frac{m\pi}{l}\right)x$

Now $\dot{u}(x,0) = 0 \Rightarrow \dot{T}(0) = 0$

Now $T(t) = D\cos(c\lambda)t + E\sin(c\lambda)t$

$\dot{T}(t) = -c\lambda D\sin(c\lambda)t + Ec\lambda\cos(c\lambda)t$

$\dot{T}(0) = 0 \Rightarrow E = 0$

So $T(t) = D\cos(c\lambda)t$

$\Rightarrow T_m(t) = D\cos\left(\frac{cm\pi}{l}\right)t$

Hence

$u_m(x,t) = \left(B\sin\frac{m\pi}{l}x\right)\left(D\cos\frac{cm\pi}{l}t\right)$

$= BD\sin\left(\frac{m\pi}{l}x\right)\cos\left(\frac{cm\pi}{l}t\right)$

So $u_n(x,t) = BD\sin\left(\frac{m\pi}{l}x\right)\cos\left(\frac{cm\pi}{l}t\right)$

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$\Rightarrow U_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{cn\pi t}{l}\right)$

$n = 1, 2, 3 \ldots\ldots$

Now by principle of superposition

$u(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$

$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$

$u(x,0) = \varphi(x)$

$\Rightarrow \varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$

$\Rightarrow \int_0^l \varphi(x) \sin\left(\frac{m\pi x}{l}\right) dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx$

$= A_n\left(\frac{l}{2}\right)$ for $m = n$

$\because \int_0^l \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0 & m \neq n \\ l/2 & m = n \end{cases}$

Hence $A_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) dx$

So general solution is given by

$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) dx\right] \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$

D'ALEMBERT'S Solution OF WAVE Equation

(91, 94)

The wave equation is given by

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$

with initial conditions

$u(x,0) = f(x)$ $\quad U_t(x,0) = g(x)$

and with boundary conditions

$u(0,t) = 0$, $u(a,t) = 0$

Let the variables w and z be defined as

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$w = x+ct$, $z = x-ct$

Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}$

$= \frac{\partial u}{\partial w}(1) + \frac{\partial u}{\partial z}(1)$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}$

Now $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} \rightarrow *$

Now $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

$= c\frac{\partial u}{\partial w} - c\frac{\partial u}{\partial z}$

$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial w} - \frac{\partial}{\partial z}$

So $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial w} - \frac{\partial u}{\partial z}\right)$

$= \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} \rightarrow **$

Using $*$ & $**$ eq. ① becomes

$\frac{\partial^2 u}{\partial w \partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} = \phi(z)$

$\Rightarrow u = \int \phi(z)\,dz + \theta(w)$

$\Rightarrow u = \varphi(w) + \psi(z)$ where $\psi(z) = \int \phi(z)dz$

$\Rightarrow u = \varphi(x+ct) + \psi(x-ct)$

Which is the most general form of the solution of wave equation called D'Alembert's solution.

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Q. Let $u(x,t)$ be the solution of the equation

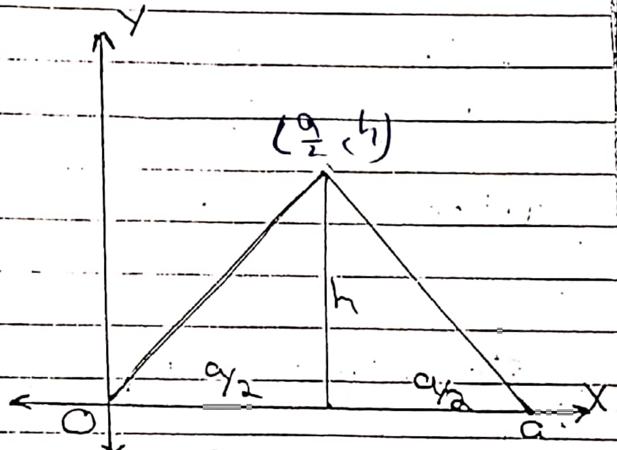
$$u_{xx} = \frac{1}{c^2} u_{tt} \qquad 0<x<a, \quad t>0$$

$u(x,0) = f(x)$, $\quad u_t(x,0) = 0$

$u(0,t) = 0$ $\quad u(a,t) = 0$

Where $f(x)$ is a function whose graph is an isosceles triangle of width 'a' and height 'h'. Find $u(x,t)$ for $x = 0.25a$, $0.5a$ and for $t = 0$, $0.2a/c$, $0.4a/c$, $0.8a/c$

Solution: The wave equation is given by



$u_{xx} = \frac{1}{c^2} u_{tt}$

with initial conditions

$$u(x,0) = f(x) = \begin{cases} \frac{2hx}{a} & 0<x<a/2 \\ \frac{-2h(x-a)}{a} & a/2<x\le a \end{cases}$$

and

$u_t(x,0) = g(x) = 0$

and with boundary conditions

$u(0,t) = 0$ $\quad u(a,t) = 0$

Now D'Alembert's general solution is given by

$u(x,t) = \varphi(x+ct) + \psi(x-ct)$

Now $u(x,0) = f(x)$

$\Rightarrow \varphi(x) + \psi(x) = f(x) \longrightarrow (1)$

$u_t(x,0) = 0 \Rightarrow \varphi'(x) - \psi'(x) = 0$

$\Rightarrow \varphi(x) - \psi(x) = A \longrightarrow (2)$

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Adding ① & ②

$\phi(x) = \frac{1}{2}\{f(x) + A\}$

Subtracting ② from ①

$\psi(x) = \frac{1}{2}\{f(x) - A\}$

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Hence

$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\}$

Now

$u(.25a, 0) = \frac{1}{2}\{f(.25a) + f(.25a)\}$

$= f(.25a) = \frac{2h(.25a)}{a}$

$= h/2$

$u(.25a, .2a/c) = h/2$

$u(.5a, .4a/c) = h/5$

$u(.5a, .8a/c) = \frac{1}{2}[f(.5a + .8a) + f(.5a - .8a)]$

$= \frac{1}{2}\{f(1.3a) + f(-.3a)\}$

Solution does not exist.

Q. Solve $u_{xx} - \frac{1}{c^2}u_{tt} = -\frac{1}{T}F(x,t)$
if $F(x,t) = T\cos t$

Find general solution of this equation.

Solution.

By D'Alembert's method

$4\frac{\partial^2 u}{\partial w \partial z} = -\frac{1}{T}F(x,t)$ where $w = x+ct$, $z = x-ct$

$\Rightarrow \frac{\partial^2 u}{\partial w \partial z} = -\frac{1}{4T} \cdot T\cos t$

$\Rightarrow \frac{\partial^2 u}{\partial w \partial z} = -\frac{1}{4}\cos\left(\frac{w-z}{2c}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial z \partial w} = -\frac{1}{4}\cos\left(\frac{w-z}{2c}\right)$

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$\Rightarrow \frac{\partial u}{\partial w} = -\frac{1}{i_1} \sin\left(\frac{w-z}{2c}\right)(-2c) + \varphi(w)$

$= \frac{1}{2} c \sin\left(\frac{w-z}{2c}\right) + \varphi(w)$

$\Rightarrow u(x,t) = \frac{1}{2} c \left[-\cos\left(\frac{w-z}{2c}\right)(+2c)\right] + \int \varphi(w)\, dw$
$+ \theta(z)$

$\Rightarrow u(x,t) = -c^2 \cos t + \psi(x+ct) + \theta(x-ct)$

Q. $u_{xx} = \alpha^2 u_t \rightarrow (1)$

with $u(0,t) = T_1 \rightarrow (2)$

$u(l,t) = T_2 \rightarrow (3)$ & $u(x,0) = \varphi(x) \rightarrow (4)$

Solution.

Define a function

$V(x,t) = u(x,t) - \left(1 - \frac{x}{l}\right) T_1 - \frac{x}{l} T_2$

Then $V(0,t) = u(0,t) - \left(1 - \frac{0}{l}\right) T_1 - \frac{0}{l} T_2$

$= T_1 - T_1 = 0$

$V(l,t) = u(l,t) - (1-1) T_1 - T_2 = T_2 - T_2 = 0$

Also $V(x,0) = u(x,0) - \left(1 - \frac{x}{l}\right) T_1 - \frac{x}{l} T_2$

$= \varphi(x) - \left(1 - \frac{x}{l}\right) T_1 - \frac{x}{l} T_2$

$\Rightarrow V(x,0) = \psi(x)$

So given problem reduces to

$V_{xx} = \alpha^2 V_t$ with

$V(0,t) = 0, \quad V(l,t) = 0, \quad V(x,0) = \psi(x)$

Let $V(x,t) = X(x) T(t)$

$X_{xx} T = \alpha^2 X T_t$

$\Rightarrow \frac{X_{xx}}{X} = \alpha^2 \frac{T_t}{T} = -\lambda^2$

$\Rightarrow \frac{X_{xx}}{X} = -\lambda^2 \Rightarrow X_{xx} + \lambda^2 X = 0$

A.E. is $D^2 + \lambda^2 = 0 \Rightarrow D = \pm i\lambda$

So $X(x) = A \cos \lambda x + B \sin \lambda x$

Also $\frac{\alpha^2 T_t}{T} = -\lambda^2 \Rightarrow \ln T = -\frac{\lambda^2}{\alpha^2} t + \ln c$

$\Rightarrow T(t) = c\, e^{-\frac{\lambda^2}{\alpha^2} t}$

$V(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$

$V(l,t) = 0 \Rightarrow X(l) = 0 \Rightarrow \lambda = \frac{n\pi}{l}$

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So $X(x) = B \sin\frac{n\pi x}{l}$, $T(t) = C e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$

$\Rightarrow v(x,t) = BC \sin\frac{n\pi x}{l} e^{-(n^2\pi^2/\alpha^2 l^2)t}$

$\Rightarrow v_n(x,t) = \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-(n^2\pi^2/\alpha^2 l^2)t}$

$v(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-(n^2\pi^2/\alpha^2 l^2)t}$

$v(x,0) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)$

$\Rightarrow \psi(x) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right)$

$\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx = \sum_{n=1}^{\infty} \lambda_n \int_0^{l} \sin\frac{m\pi x}{l} \times \sin\frac{n\pi x}{l} dx$ $\neq 0$

$\Rightarrow \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx = \lambda_n\left(\frac{l}{2}\right)$ for $m=n$

$\Rightarrow \lambda_n = \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$

So $v(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx\right) \times \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$

Thus

$u(x,t) = \left(1-\frac{x}{l}\right)T_1 = \frac{x}{l}T_2$

$+ \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx\right] \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$

Where $\psi(x) = \phi(x) - \left(1-\frac{x}{l}\right)T_1 - \frac{x}{l}T_2$

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Q:- $U_{xx} = \alpha^2 U_t$ 91^R, 91

with $u_x(0,t)=0$

$u_x(l,t)=0$ & $u(x,0) = \phi(x)$

Sol. Let $u(x,t) = X(x)T(t)$

$X(x) = A\cos\lambda x + B\sin\lambda x$

$T(t) = Ce^{-\lambda^2/\alpha^2 t}$

$X_x(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X_x(0) = 0 \Rightarrow B=0$

$X_x(l) = 0 \Rightarrow -A\lambda\sin\lambda x = 0$

$\Rightarrow \lambda = \frac{n\pi}{l}$ $n \in W$.

$\Rightarrow u(x,t) = AC\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$

$\Rightarrow u_n(x,t) = \lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$

So by principle of super position

$$u(x,t) = \sum_{n=0}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

$$= \lambda_0 + \sum_{n=1}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

$u(x,0) = \phi(x)$

$\Rightarrow \phi(x) = \lambda_0 + \sum_{n=1}^{\infty}\lambda_n\cos\frac{n\pi x}{l}$

$(\phi(x) - \lambda_0) = \sum_{n=1}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)$

$\int_0^l(\phi(x) - \lambda_0)\cos\frac{n\pi x}{l}\,dx = \lambda_n\left(\frac{l}{2}\right)$

$\Rightarrow \lambda_n = \frac{2}{l}\int_0^l(\phi(x) - \lambda_0)\cos\left(\frac{n\pi x}{l}\right)dx$

Hence

$u(x,t) = \lambda_0 + \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l(\phi(x) - \lambda_0)\cos\left(\frac{n\pi x}{l}\right)dx\right] \times \cos\left(\frac{n\pi x}{l}\right)dx$

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Q:- $u_{xx} = \alpha^2 u_t$
with $u(0,t) = T_0$ $u_x(l,t) = 0$
& $u(x,0) = \phi(x)$

Sol:-
Define a function
$V(x,t) = u(x,t) - T_0$
Then given problem becomes
$V_{xx} = \alpha^2 V_t$ with
$V(0,t) = 0$ $V_x(l,t) = 0$
$V(x,0) = u(x,0) - T_0$
$= \varphi(x) - T_0 = \psi(x)$

Let $V(x,t) = X(x)T(t)$
Then $X(x) = A\cos\lambda x + B\sin\lambda x$
& $T(t) = e^{(-\lambda^2/\alpha^2)t}$
$X(0) = 0 \Rightarrow A = 0$
So $X(x) = B\sin\lambda x$
$X_x(x) = B\lambda\cos\lambda x$
$X_x(l) = 0 \Rightarrow B\lambda\cos\lambda l = 0$
$\Rightarrow \lambda l = (2n+1)\cdot\frac{\pi}{2}$
$\Rightarrow \lambda = (2n+1)\frac{\pi}{2l}$

So $V_n(x,t) = \lambda_n \sin\left(\frac{(2n+1)\pi x}{2l}\right) e^{-\frac{(2n+1)^2\pi^2}{4l^2\alpha^2}t}$

So by principle of superposition
$$V(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{(2n+1)\pi x}{2l}\right) e^{-\frac{(2n+1)^2\pi^2}{4l^2\alpha^2}t}$$

Where $\lambda_n = \frac{2}{l}\int_0^l (\varphi(x) - T_0)\sin\left(\frac{(2n+1)\pi}{2l}x\right)dx$

So $u(x,t) = T_0 + \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{(2n+1)\pi x}{2l}\right) e^{\frac{-(2n+1)^2\pi^2 t}{4\alpha^2 l^2}}$

where $\lambda_n = \frac{2}{l}\int_0^l (\varphi(x) - T_0)\sin\left(\frac{(2n+1)\pi x}{2l}\right)dx$

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Q:- $U_{xx} = \alpha^2 U_t$

with $U(0,t)=0$, $U(l,t)=T_0$

& $U(x,0) = \phi(x)$

Sol.

Define a function

$V(x,t) = U(x,t) - \frac{x}{l} T_0$

Then given problem reduces to

$V_{xx} = \alpha^2 V_t$ with $V(0,t)=0$

$V(l,t)=0$, $V(x,0) = \varphi(x) - \frac{x}{l} T_0 = \psi(x)$

Let $V(x,t) = X(x)\,T(t)$

$\Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$

& $T(t) = e^{-\frac{\lambda^2}{\alpha^2}t}$

$V(0,t)=0 \Rightarrow X(0)=0 \Rightarrow A=0$

$V(l,t)=0 \Rightarrow X(l)=0 \Rightarrow B\sin\lambda l=0$

$\Rightarrow \lambda = \frac{n\pi}{l}$

So

$$V(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

Where $\lambda_n = \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$

So

$$U(x,t) = \frac{x}{l}T_0 + \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

Where

$$\lambda_n = \frac{2}{l}\int_0^l \left(\varphi(x) - \frac{x}{l}T_0\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

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Q. Solve
$u_{xx} = \alpha^2 u_t$
with $u(0,t) = T_1$
$u(l,t) = T_2$
& $u(x,0) = 0$

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Sol.
Define a function
$V(x,t) = u(x,t) - (1-\frac{x}{l})T_1 - \frac{x}{l}T_2$
Then
$V(0,t) = u(0,t) - T_1 - 0$
$= T_1 - T_1 - 0$
$\Rightarrow V(0,t) = 0$
$V(l,t) = u(l,t) - 0 - T_2$
$= T_2 - T_2$
$\Rightarrow V(l,t) = 0$
Then the given problem reduces to
$V_{xx} = \alpha^2 V_t$
with $V(0,t) = 0$, $V(l,t) = 0$
& $V(x,0) = -(1-\frac{x}{l})T_1 - \frac{x}{l}T_2$
Now as
Put $V = XT$
$\Rightarrow V = X(x)T(t)$
$\Rightarrow X_{xx}T = \alpha^2 X T_t$
$\Rightarrow \frac{X_{xx}}{X} = \alpha^2 \frac{T_t}{T} = -\lambda^2$
$\Rightarrow X_{xx} + \lambda^2 X = 0$
$\Rightarrow$ A.E is $D^2 + \lambda^2 = 0$
$\Rightarrow D^2 = -\lambda^2$
$\Rightarrow D = \pm i\lambda$
so $X(x) = A\cos\lambda x + B\sin\lambda x$
Also $\frac{T_t}{T} = \frac{-\lambda^2}{\alpha^2}$

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$\Rightarrow T(t) = C e^{-\frac{\lambda^2}{\alpha^2} t}$

So $V(x,t) = (A\cos\lambda x + B\sin\lambda x)\, C e^{-\frac{\lambda^2}{\alpha^2} t}$

$V(0,t) = 0 \Rightarrow X(0) = 0$

$\Rightarrow A\cos(0) + B\sin(0) = 0$

$\Rightarrow A = 0$

So $X(x) = B\sin\lambda x$

$X(l) = 0 \Rightarrow B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$

$\Rightarrow V(x,t) = B\sin\left(\frac{n\pi x}{l}\right) \cdot C e^{-\frac{n^2\pi^2}{\alpha^2 l^2} t}$

$\Rightarrow V_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2} t}$

Now by principle of superposition

$V(x,t) = \sum_{n=1}^{\infty} V_n(x,t)$

$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2} t}$

$V(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) (1)$

$\Rightarrow \left(\frac{x}{l}T_1 - \frac{x}{l}T_2 - T_1\right) = \sum_{n=1} A_n \sin\left(\frac{n\pi x}{l}\right)$

$\left(\frac{x}{l}T_1 - \frac{x}{l}T_2 - T_1\right) \sin\frac{m\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right)$

$\Rightarrow \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \int_0^l \left(\frac{x}{l}T_1 - \frac{x}{l}T_2 - T_1\right) \sin\frac{m\pi x}{l}\, dx$

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For $m = n$

$$A_n\left(\frac{l}{2}\right) = \int_0^l \left(\frac{x}{l}T_1 - \frac{x}{l}(T_2 - T_1)\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \left(\frac{T_1 - T_2}{l}\right)\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx - T_1\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \left(\frac{T_1 - T_2}{l}\right)\left\{\frac{-x\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\Bigg|_0^l - \int_0^l (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right\}$$

$$- T_1\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]\Bigg|_0^l$$

$$= \left(\frac{T_1 - T_2}{l}\right)\left\{\frac{-l\cos n\pi}{n\pi/l} - 0 + \int_0^l \frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right\}$$

$$+ \frac{lT_1}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \left(\frac{T_1 - T_2}{l}\right)\left[-\frac{l^2}{n\pi}(-1)^n + \frac{l}{n\pi}\cdot\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\Bigg|_0^l\right]$$

$$+ \frac{lT_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \left(\frac{T_1 - T_2}{l}\right)\left[-\frac{l^2}{n\pi}(-1)^n + \frac{l^2}{n^2\pi^2}\left[0 - 0\right]\right]$$

$$+ \frac{lT_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= -\frac{(T_1 - T_2)l}{n\pi}(-1)^n + \frac{lT_1}{n\pi}(-1)^n - \frac{lT_1}{n\pi}$$

$$= -\frac{lT_1(-1)^n}{n\pi} + \frac{lT_2}{n\pi}(-1)^n + \frac{lT_1}{n\pi}(-1)^n - \frac{lT_1}{n\pi}$$

$$\Rightarrow \lambda_n \left(-- \right) = \frac{l T_2}{n\pi}(-1)^n - \frac{l T_1}{n\pi}$$

$$\Rightarrow \lambda_n = \frac{2}{l}\left[\frac{l}{n\pi}\left(T_2(-1)^n - T_1 \right) \right]$$

$$\Rightarrow \lambda_n = \frac{2}{n\pi}\left[(-1)^n T_2 - T_1 \right]$$

Hence

$$v(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(-1)^n T_2 - T_1 \right] \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

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HEAT EQUATION IN POLAR FORM

Heat equation in two dimension is given by

$$u_{xx} + u_{yy} = \frac{1}{k} u_t$$

Where $u = u(x,y,t)$.

We can also write the above equation as

$$\nabla^2 u = \frac{1}{k} u_t$$

In polar from it can be written as

$$\nabla^2 u(r,\theta,t) = \frac{1}{k}\frac{\partial}{\partial t} u(r,\theta,t) \rightarrow ①$$

subject to the conditions

$$u(a,\theta,t) = f(\theta) \rightarrow ②$$

$$\& \; u(r,\theta,0) = g(r,\theta) \rightarrow ③$$

This is a non homogeneous problem. To make it homogeneous. 1st we obtain its steady state solution $V(r,\theta)$.

For the purpose put

$$\lim_{t\to\infty} u(r,\theta,t) = V(r,\theta) \quad -\pi<\theta<\pi$$

Then the problem for steady state reduces to

$$\nabla^2 V(r,\theta) = 0$$

with $V(a,\theta) = f(\theta) \quad -\pi<\theta<\pi$

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The steady state solution can be obtained after solving the above problem. The transient state solution $w(r,\theta,t)$ is given by

$$w(r,\theta,t) = u(r,\theta,t) - V(r,\theta)$$

Where $V(r,\theta)$ is the steady state solution. Then ①, ② and ③ takes the form

$$\nabla^2 w(r,\theta,t) = \frac{1}{k}\frac{\partial}{\partial t} w(r,\theta,t) \rightarrow ④$$

with

$$w(a,\theta,t) = 0$$

$$\& \quad w(r,\theta,0) = g(r,\theta) - V(r,\theta) \rightarrow ⑤$$

Now we have to solve boundary value problem defined by ④ and ⑤ by using the method of seperation of variables.

Let

$$w(r,\theta,t) = \phi(r,\theta)\,\psi(t).$$

Then by ④

$$\nabla^2(\phi\psi) = \frac{1}{k}\frac{\partial}{\partial t}(\phi\psi)$$

$$\Rightarrow \psi\,\nabla^2\phi = \frac{1}{k}\,\phi\frac{\partial}{\partial t}\psi$$

Dividing both sides by $\phi\psi$

$$\frac{\nabla^2\phi}{\phi} = \frac{1}{k}\frac{\psi'}{\psi}.$$

Let $\dfrac{\nabla^2\phi}{\phi} = \dfrac{1}{k}\dfrac{\psi'}{\psi} = -\lambda^2$

$$\Rightarrow \nabla^2\phi + \lambda^2\phi = 0 \rightarrow ⑥$$

$$\& \quad \frac{\psi'}{\psi} = -\lambda^2 k \rightarrow ⑦$$

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From ⑦

$$\frac{\psi'}{\psi} = -\lambda^2 k$$

$$\Rightarrow \ln\psi = -\lambda^2 kt + \ln c$$

$$\Rightarrow \ln\psi = \ln c e^{-\lambda^2 kt}$$

$$\Rightarrow \psi = c e^{-\lambda^2 kt} \rightarrow ⑧$$

Now from ⑥

$$\nabla^2\varphi + \lambda^2\varphi = 0$$

$$\Rightarrow \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)\right]\varphi + \lambda^2\varphi = 0$$

$$\because \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)$$

So $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \lambda^2\varphi = 0 \rightarrow ⑨$

Now put $\varphi(r,\theta) = P(\theta)R(r)$

Then by ⑨

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)P + \frac{1}{r^2}\frac{\partial^2 P}{\partial\theta^2}\cdot R + PR\lambda^2 = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)P + \frac{1}{r^2}\frac{\partial^2 P}{\partial\theta^2}R = -\lambda^2 PR$$

Dividing both sides by PR

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2 P}\frac{\partial^2 P}{\partial\theta^2} = -\lambda^2$$

$$\Rightarrow \frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2 P}\frac{d^2 P}{d\theta^2} = -\lambda^2$$

$$\Rightarrow \frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2 P}P'' = -\lambda^2$$

$$\Rightarrow \frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2}\frac{P''}{P} = -\lambda^2 \rightarrow (10)$$

Now here P''/P must be constant, otherwise λ^2 will not be constant.

So take $P''/P = -\mu^2 \rightarrow (11)$

Then $P'' + \mu^2 P = 0$

A.E. is $D^2 + \mu^2 = 0$

$\Rightarrow D = \pm i\mu$

So $P(\theta) = c_1 \cos\mu\theta + c_2 \sin\mu\theta \rightarrow (12)$

By using this substitution (10) takes the form

$$\frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2}(-\mu^2) = -\lambda^2$$

Xing both sides by $r^2 R$

$$r\frac{d}{dr}(rR') + (-\mu^2)R = -\lambda^2 r^2 R$$

$$\Rightarrow r\,[rR'' + R'] - \mu^2 R = -\lambda^2 r^2 R$$

$$\Rightarrow (r^2 R'' + rR') + (\lambda^2 r^2 - \mu^2)R = 0 \rightarrow (13) \quad 0<r<a$$

Which is well known Bessel's Differential equation. Which can be solved by using the method of series solution.

For this put

$$R = \sum_{k=0}^{\infty} C_k r^{p+k}$$

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Then after further solving we get

$$C_k = \frac{\lambda^2}{\mu^2 - (\mu+k)} C_{k-2} \quad k \geq 2$$

$$C_k = \frac{\lambda^2}{k(k+2\mu)} C_{k-2} \quad k \geq 2 \longrightarrow (14)$$

Which is recurrance formula and it shows that $C_2, C_4, C_6, \ldots, C_{2m}$ can be expressed in terms of C_0 and $C_3, C_5, C_7, \ldots, C_{2m+1}$ can be expressed in terms of C_1. But as $C_1 = 0$, so

$C_3 = 0, C_5 = 0, C_7 = 0, \ldots, C_{2m+1} = 0$

Putting values of k we finally get from (14)

$$C_{2m} = \frac{(\lambda/2)^{2m} (-1)^m C_0}{m! (\mu+1)(\mu+2)\cdots(\mu+m)}$$

Now $R(r) = \sum_{k=0}^{\infty} C_k r^{p+k}$

$$= \sum_{m=0}^{\infty} C_{2m} r^{\mu+2m}$$

$$\Rightarrow R(r) = \sum_{m=0}^{\infty} \frac{(\lambda/2)^{2m} r^{2m+\mu} (-1)^m (\lambda/2)^{\mu}}{m! (\mu+1)(\mu+2)\cdots(\mu+m) \cdot \mu!}$$

$$\because C_0 = \frac{(\lambda/2)^{\mu}}{\mu!}$$

$$\Rightarrow R(r) = \left(\frac{\lambda r}{2}\right)^{\mu} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (\mu+m)!} \left(\frac{\lambda r}{2}\right)^{2m}$$

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This solution of Bessel's Equation is called Bessel's Function.

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Eigen Values & eigen functions.

Suppose we are given a set of equations involving a parameter λ and we are required to find the value of λ for which the system has non trivial solution. Such a problem is called an eigen value problem, and the values of λ for which solution is non-trivial are called eigen values and the corresponding ~~...~~ solutions are called eigen functions.

SPECTRUM.

The set of all eigenvalues of a problem is called spectrum.

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STURM-LIOUVILLE SYSTEMS

Jacques Charles Francois Sturm (1803-1855) was borne in switzerland and then moved to Paris.
Joseph Liouville (1809-1882) was a professor in paris and worked in Complex Analysis, diff. geometry & no. Theory.

Sturm Liouville Equation:

A 2nd order linear homogeneous differential equation of the form

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}+q(x)y(x)+\lambda r(x)\,y=0$$

is called sturm Liouville equation if $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are all real and continuous over an interval $[a,b]$.

If we define an operator L as

$$L=\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\}+q(x)$$

Then above eq. takes the form

$$L(y)+\lambda r(x)\,y=0$$

Examples:

(i) The Legendre diff. eq.

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\}+n(n+1)u=0$$

is S.L. Equation with

$p(x)=1-x^2$ $\quad q(x)=0$ $\quad r(x)=1$ and

$\lambda=n(n+1)$.

(ii) The Airy's diff. eq.

$$\frac{d^2y}{dx^2}+xy=0 \qquad x\in\mathbb{R}$$

is also an S.L. eq. Because it can be written as

$$\frac{d}{dx}\left\{1\cdot\frac{dy}{dx}\right\}+0\cdot y+1\cdot xy=0$$

with $p(x)=1$ $\quad q(x)=0$ $\quad r(x)=x$

and $\lambda=1$

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S. L. System :-

An S. L. Eq. with some suitable initial and boundry conditions is called S. L. System.

Regular S. L. Equation:

The S. L. Equation is said to be regular over the interval [a, b] if the coefficients p(x) and r(x) do not vanish over this interval.

Examples:

1) The diff. eq.

$u'' + \lambda u = 0$ is regular S. L. eq. Because here $p(x) = 1$ and $r(x) = 1$ Which donot vanish every where.

2) The Legender diff. eq.

$\frac{d}{dx}\{(1-x^2)\frac{du}{dx}\} + n(n+1)u = 0$ is not regular in any interval containing ± 1. Because here $p(x) = 1 - x^2$ which vanishes in any interval containing ± 1.

S. L. System: (Regular).

A regular S. L. equation with suitable boundry conditions is called regular S. L. System.

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Singular S.L. Equation:

The S.L. Eq. which is not regular is said to be singular in the interval. i.e.

The S.L. Eq. is said to be singular in the interval $[a,b]$, if $p(x)$ or $r(x)$ vanishes at any point in the interval. e.g. The Legender's diff. eq. is singular in any interval containing ± 1

Regular S.L. System.

The regular S.L. Equation $\frac{d}{dx}\{p(x)\frac{du}{dx}\} + q(x)u + \lambda r(x)u = 0$ forms regular S.L. System over the interval $[a,b]$ if u satisfies the end point conditions

$$\alpha u(a) + \alpha' u'(a) = 0$$

$$\beta u(b) + \beta' u'(b) = 0$$

e.g

(i) $u'' + \lambda u = 0$ defines regular S.L. system with $u(a)=0$ $u(b)=0$

(ii) $u'' + \lambda u = 0$

$u'(a) = 0$ $u'(b) = 0$

is also Regular S.L. System.

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Periodic Function.

A function f is said to be periodic function of period λ if $f(x) = f(x+\lambda)$, where λ is the least for which this condition holds.

Periodic S.L. Equation.

The S.L. eq.

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda r(x)u = 0$$

is said to be periodic S.L. equation if $p(x)$, $q(x)$, and $r(x)$ are all the periodic functions of period $b-a$. i.e

$$p(a) = p(b), \quad q(a) = q(b), \quad r(a) = r(b)$$

Periodic S.L. System.

The periodic S.L. eq. with end point conditions of the form

$$u(a) = u(b) \quad \text{and} \quad u'(a) = u'(b)$$

defines the periodic S.L. System.

Example: The Mathieu's Diff. Eq.

$$u'' + (\lambda + 16d\cos 2x)u = 0$$

with end point conditions

$$u(-\pi) = u(\pi) \quad \text{and} \quad u'(-\pi) = u'(\pi)$$

is a periodic S.L. System with

$p(x) = 1$, $q(x) = 16d\cos 2x$, $r(x) = 1$

and it period is 2π

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Eigenvalue & Eigensolution of an S.L. System.

If in an S.L. System (Regular or Periodic) corresponding to the value λ there is a non zero solution $u(x)$, Then $u(x)$ is called eigensolution or eigenfunction and the value of λ is called eigenvalue of the system.

e.g. The system
$$u'' + \lambda u = 0 \quad \text{with}$$
$$u(0) = 1 \quad \text{and} \quad u'(0) = -2$$
Then for $\lambda = 4$
$$u'' + 4u = 0 \Rightarrow u = c_1 \sin 2x + c_2 \cos 2x$$
$$u(0) = 1 \Rightarrow c_2 = 1$$
$$\& \; u'(0) = -2 \Rightarrow 2c_1 = -2 \Rightarrow c_1 = -1$$
$$\Rightarrow u = -1 \cdot \sin 2x + 1 \cdot \cos 2x$$
$$\Rightarrow u = \cos 2x - \sin 2x$$
Hence here $\lambda = 4$ is the eigen value and $u = \cos 2x - \sin 2x$ is the corresponding eigen solution.

EXAMPLE (1)

Find eigenvalues and eigenfunctions of the regular S.L. System
$$u'' + \lambda u = 0$$
with $u(0) = 0$ and $u(\pi) = 0$

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SOLUTION :-

Given $u'' + \lambda u = 0$

$\Rightarrow \frac{d}{dx}\{1 \cdot \frac{du}{dx}\} + 0 \cdot u + \lambda(1) u = 0$

$u(0) = 0 \qquad u(\pi) = 0$

Here $p(x) = 1 \qquad q(x) = 0 \qquad r(x) = 1$

$a = 0 \qquad b = \pi$

Now A.E. is $D^2 + \lambda = 0$

$\Rightarrow D = \pm i\sqrt{\lambda}$

$\Rightarrow u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$0 \leq x < \pi$

Now $u(0) = 0 \Rightarrow A = 0$

$\Rightarrow u(x) = B\sin\sqrt{\lambda}\,x$, $u(\pi) = 0 \Rightarrow B\sin\sqrt{\lambda}\,\pi = 0$

$\Rightarrow$ either $B = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$

But $B \neq 0$ because then $u(x) = 0$ which is trivial solution.

So $\sin(\sqrt{\lambda}\,\pi) = 0$

$\Rightarrow \sqrt{\lambda}\,\pi = n\pi \qquad n = \pm 1, \pm 2, \pm 3.$

Here $n \neq 0 \quad \because$ if $n = 0$ then $u = 0$

Hence $\sqrt{\lambda} = n$

$\Rightarrow \lambda = n^2 \qquad n = 1, 2, 3, \ldots$

$\Rightarrow \lambda = \lambda_n \qquad n = 1, 2, 3, \ldots$

Hence for these values of λ non trivial solution or solutions can be found. So these values are the eigenvalues of the diff. equation. And the corresponding eigen solutions are

$u_n = b_n \sin nx \qquad n = 1, 2, 3, \ldots$

Where the constants b_n are in general different for each solution

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Example 2:-

Show that if $u(x)$ and $v(x)$ are the periodic solutions of the Mathieu's diff. eq.

$$u'' + \lambda u + 16d\cos 2x\, u = 0$$

with period π having distinct eigen values then $\int_a^b u(x)\,v(x)\,dx = 0$

Solution:-

Here the end point conditions are $u(0) = u(\pi)$ and $u'(0) = u'(\pi)$

Let λ_1, λ_2 be the two distinct eigen values corresponding to the eigenfunctions $u(x)$ and $v(x)$ respectively.

Then

$$u'' + 16d\cos 2x\, u + \lambda_1 u = 0 \rightarrow (1) \text{ with } u(0) = u(\pi) \;\&\; u'(0) = u'(\pi)$$

and $v'' + 16d\cos 2x\, v + \lambda_2 v = 0 \rightarrow (2)$

with $v(0) = v(\pi)$ and $v'(0) = v'(\pi)$

Now Xing eq. (1) by v and eq. (2) by u and then subtracting

$$u''v + 16d\cos 2x\, uv + \lambda_1 uv - uv'' - 16d\cos 2x\, uv - \lambda_2 uv = 0$$

$$u''v - uv'' = (\lambda_2 - \lambda_1)\,uv$$

$$\Rightarrow (\lambda_2 - \lambda_1)\,uv = \frac{d}{dx}(u'v - uv')$$

Integrating both sides from 0 to π

$$(\lambda_2 - \lambda_1)\int_0^{\pi} u(x)v(x)\,dx = u'(x)v(x) - u(x)v'(x)\Big|_0^{\pi}$$

$$(\lambda_2 - \lambda_1)\int_0^{\pi} u(x)v(x)\,dx = u'(\pi)v(\pi) - u(\pi)v'(\pi) - u'(0)v(0) + u(0)v'(0)$$

$$= u'(\pi)v(\pi) - u'(\pi)v'(\pi) - u'(0)v'(0) + u'(0)v'(0)$$

$$= 0 \quad \text{By using end point conditions}$$

$$\Rightarrow \int_0^{\pi} u(x)v(x)\,dx = 0 \quad \because \lambda_2 - \lambda_1 \neq 0 \quad \text{As } \lambda_1 \neq \lambda_2$$

EXAMPLE 3:-

Determine eigenvalues of the system

$$u'' + \lambda u = 0$$

with $u(0) = u(\pi)$, $u'(0) = 2u'(\pi)$

Solution:-

A.E. is $D^2 + \lambda = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

$\Rightarrow u = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$u'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$u(0) = u(\pi)$

$\Rightarrow A = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

$B\sqrt{\lambda} = (-2A\sin\sqrt{\lambda}\pi + 2B\cos\sqrt{\lambda}\pi)\sqrt{\lambda}$

$\Rightarrow (1-\cos\sqrt{\lambda}\pi)A - \sin\sqrt{\lambda}\pi \cdot B = 0$

$2A\sin\sqrt{\lambda}\pi + (B - 2B\cos\sqrt{\lambda}\pi) = 0$

For non trivial solution for A and B

$$\begin{vmatrix} 1-\cos\sqrt{\lambda}\pi & -\sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1-2\cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$\Rightarrow (1-\cos(\sqrt{\lambda}\pi))(1-2\cos(\sqrt{\lambda}\pi)) + 2\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\pi) = 0$

~~$\Rightarrow 1 - 2\sin(\sqrt{\lambda}\pi) - \cos\sqrt{\lambda}\pi + 2\sin\sqrt{\lambda}\pi\cos\sqrt{\lambda}\pi + 2\sin\sqrt{\lambda}\pi\cos\sqrt{\lambda}\pi$~~

$\Rightarrow 1 - 2\cos\sqrt{\lambda}\pi - \cos\sqrt{\lambda}\pi + 2\cos^2\sqrt{\lambda}\pi + 2\sin^2\sqrt{\lambda}\pi = 0$

$\Rightarrow 1 - 3\cos\sqrt{\lambda}\pi + 2(\cos^2\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi) = 0$

$\Rightarrow 1 - 3\cos\sqrt{\lambda}\pi + 2(1) = 0$

$\Rightarrow 3 - 3\cos\sqrt{\lambda}\pi = 0 \Rightarrow 3\cos\sqrt{\lambda}\pi = 3$

$\Rightarrow \cos\sqrt{\lambda}\pi = 1$

$\Rightarrow \sqrt{\lambda}\pi = 2n\pi$ $\quad n = 0, \pm1, \pm2, \pm3, \ldots$

$\Rightarrow \lambda = 4n^2$ $\quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \lambda_n = 4n^2$ $\quad n = 0, 1, 2, 3, \ldots$

These are the eigenvalues and the corresponding eigen functions are

$$u_n(x) = A_n\cos(2nx) + B_n\sin(2nx)$$

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ORTHOGONALITY of EIGENFUNCTIONS OF AN S.L. SYSTEM

Definition:

Two Eigenfunctions $u(x)$ and $v(x)$ are said to be orthogonal w.r.t. a weight function $w(x)$ over $[a,b]$ if

$$\int_a^b w(x)\, u(x)\, v(x)\, dx = 0$$

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Orthogonality of Solutions of a regular S.L. System.

If $u_m(x)$ and $u_n(x)$ are solutions of a regular S.L. System corresponding to distinct eigenvalues λ_m and λ_n of the parameter λ, then $u_m(x)$ and $u_n(x)$ are orthogonal w.r.t. the weight function $r(x)$, i.e.

$$\int_a^b u_m(x)\, u_n(x)\, r(x)\, dx = 0 \qquad m \neq n$$

PROOF:- The regular S.L. System is given by

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)\,u(x) + \lambda\, r(x)\, u(x) = 0 \qquad a \le x \le b \quad \rightarrow (1)$$

with end point conditions

$$\alpha u(a) + \alpha' u'(a) = 0$$
$$\beta u(b) + \beta' u'(b) = 0$$

Now as $u_m(x)$, $u_n(x)$ are solutions of (1) corresponding to the distinct eigenvalues λ_m and λ_n so

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$$\frac{d}{dx}\{P(x)\,u_m'(x)\} + q(x)\,u_m(x) + \lambda_m\,r(x)\,u_m(x) = 0 \longrightarrow (1)$$

with

$$\alpha u_m(a) + \alpha' u_m'(a) = 0, \quad \beta u_m(b) + \beta' u_m'(b) = 0\} \rightarrow (2)$$

and $$\frac{d}{dx}\{P(x)\,u_n'(x)\} + q(x)\,u_n(x) + \lambda_n\,r(x)\,u_n(x) = 0 \longrightarrow (3)$$

with

$$\alpha u_n(a) + \alpha' u_n'(a) = 0, \quad \beta u_n(b) + \beta' u_n'(b) = 0\} \rightarrow (4)$$

Now multiplying (1) by $u_n(x)$ and (3) by $u_m(x)$ and then subtracting

$$\frac{d}{dx}\{P(x)\,u_m'(x)\}\,u_n(x) + q(x)u_m(x)u_n(x) + \lambda_m r(x)u_m(x)u_n(x)$$
$$- \frac{d}{dx}\{P(x)\,u_n'(x)\}\,u_m(x) - q(x)u_m(x)u_n(x) - \lambda_n r(x)\,u_m(x)u_n(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left[P(x)\{u_m(x)u_n'(x) - u_m'(x)u_n(x)\}\right] = (\lambda_m - \lambda_n)\,r(x)\,u_m(x)\,u_n(x)$$

Integrating both sides w.r.t. x from a to b

$$(\lambda_m - \lambda_n)\int_a^b r(x)\,u_m(x)\,u_n(x)\,dx = \left[P(x)\{u_m(x)u_n'(x) - u_m'(x)u_n(x)\}\right]_a^b$$

$$= P(b)\{u_m(b)u_n'(b) - u_m'(b)u_n(b)\} - P(a)\{u_m(a)u_n'(a) - u_m'(a)u_n(a)\} \rightarrow (5)$$

From (2) & (3)

$$u_m'(a) = -\frac{\alpha}{\alpha'}u_m(a) \qquad u_m'(b) = -\beta/\beta'\,u_m(b)$$
$$u_n'(a) = -\frac{\alpha}{\alpha'}u_n(a) \qquad u_n'(b) = -\beta/\beta'\,u_n(b)$$

Using these values eq. (5) takes the form

$$(\lambda_m - \lambda_n)\int_a^b r(x)\,u_m(x)\,u_n(x)\,dx = P(b)\Big\{u_m(b)\left(-\beta/\beta'\,u_n(b)\right)$$
$$- \left(-\beta/\beta'\,u_m(b)\right)u_n(b)\Big\} - \Big\{P(a)\Big\{u_m(a)\left(-\alpha/\alpha'\,u_n(a)\right) + \alpha/\alpha'\,u_m(a)u_n(a)\Big\}$$

$$= P(b)\left[-\frac{\beta}{\beta'}u_m(b)u_n(b) + \beta/\beta'\,u_m(b)u_n(b)\right] + \left[\frac{\alpha}{\alpha'}u_m(a)u_n(a) - \frac{\alpha}{\alpha'}u_n(a)u_m(a)\right]P(a)$$

$$= P(a)(0) + P(b)(0) = 0$$

As $m \neq n \Rightarrow \lambda_m - \lambda_n \neq 0 \Longrightarrow \int_a^b r(x)\,u_m(x)\,u_n(x)\,dx = 0$

As req.

Orthogonality of Eigensolutions Of Periodic S.L. System

Eigensolutions corresponding to distinct eigenvalues of a periodic S.L. System are orthogonal w.r.t. the weight function $r(x)$.

Proof:-

Let $u_m(x)$ and $u_n(x)$ be the two eigensolutions corresponding to two distinct eigenvalues λ_m and λ_n of the periodic S.L. System

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda\, r(x).u(x) = 0 \qquad a \le x \le b$$

with endpoint conditions

$u(a) = u(b)$ and $u'(a) = u'(b)$

Then

$$\frac{d}{dx}\{p(x).u'_m(x)\} + q(x)\,u_m(x) + \lambda_m r(x).\,u_m(x) = 0 \longrightarrow (1)$$

with $u_m(a) = u_m(b)$ and $u'_m(a) = u'_m(b)$ $\} \longrightarrow (2)$

and $$\frac{d}{dx}\{p(x).u'_n(x)\} + q(x)\,u_n(x) + \lambda_n\, r(x).u_n(x) = 0 \longrightarrow (3)$$

with $u_n(a) = u_n(b)$ and $u'_n(a) = u'_n(b)$ $\} \longrightarrow (4)$

Xing (1) by $u_n(x)$ and (3) by $u_m(x)$ and then subtracting

$$\frac{d}{dx}\left\{p(x)\left(u_m(x).u'_n(x) - u'_m(x).u_n(x)\right)\right\} = (\lambda_m - \lambda_n)\, r(x)\, u_m(x)\, u_n(x)$$

Integrating both sides w.r.t. x from a to b

$$(\lambda_m - \lambda_n)\int_a^b r(x)u_m(x)u_n(x)\,dx = p(x)\left(u_m(x)u'_n(x) - u'_m(x)u_n(x)\right)\Big|_a^b$$

$$= p(b)\left(u_m(b)u'_n(b) - u'_m(b)\,u_n(b)\right)$$
$$- p(a)\left(u_m(a)u'_n(a) - u'_m(a)\,u_n(a)\right)$$

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Using eq. ② and ④ and the result $p(b) = p(a)$ we get

$$(\lambda_m - \lambda_n) \int_a^b r(x)\, u_m(x)\, u_n(x)\, dx = p(a)\left\{u_m(a)\, u_n'(a) - u_m'(a)\, u_n(a)\right\} - p(a)\left\{u_m(a)\, u_n'(a) - u_m'(a)\, u_n(a)\right\}$$

$$= 0$$

$$\Rightarrow (\lambda_m - \lambda_n) \int_a^b r(x)\, u_m(x)\, u_n(x)\, dx = 0$$

Since λ_m & λ_n are distinct so $\lambda_m - \lambda_n \neq 0$

$$\Rightarrow \int_a^b r(x)\, u_m(x)\, u_n(x)\, dx = 0$$ As Req.

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Reality of Eigenvalues:-

The eigenvalues of a regular S. L. System are real.

Proof:- Let the regular S. L. System be given by

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\}+q(x)\,u(x)+\lambda r(x)\,u(x)=0 \longrightarrow (1)$$

with
$$\left.\begin{aligned}\alpha u(a)+\alpha' u'(a)&=0\\ \text{and}\quad \beta u(b)+\beta' u'(b)&=0\end{aligned}\right\}\longrightarrow (2)$$

Let λ be the eigenvalues corresponding to the eigensolution $u(x)$

Let $\lambda=\lambda_1+i\lambda_2$

Since λ is supposed to be a complex so $u(x)$ is also complex. Let $u(x)=\phi(x)+i\psi(x)$

By (1)

$$\frac{d}{dx}\{p(x)(\phi'+i\psi')\}+q(x)(\phi+i\psi)+(\lambda_1+i\lambda_2)r(x)(\phi+i\psi)=0$$

Equating real and imaginary parts.

$$\frac{d}{dx}\{p(x)\phi'\}+q(x)\phi(x)+\lambda_1 r(x)\phi=0 \longrightarrow (3)$$

$$\frac{d}{dx}\{p(x)\psi'\}+q(x)\psi(x)+\lambda_2 r(x)\psi(x)=0 \longrightarrow (4)$$

Also by (2)

$$\left.\begin{aligned}\alpha\phi(a)+\alpha'\phi'(a)&=0 \qquad & \alpha\psi(a)+\alpha'\psi'(a)&=0\\ \beta\phi(b)+\beta'\phi'(b)&=0 & \beta\psi(b)+\beta'\psi'(b)&=0\end{aligned}\right\}\longrightarrow (5)$$

Multiplying eq. (3) by ψ and eq. (4) by ϕ and then subtracting

$$r(x)\lambda_2(\phi^2+\psi^2)=\frac{d}{dx}\{p(x)(\psi\phi'-\phi\psi')\}$$

Integrating both sides from a to b

$$\lambda_2\int_a^b r(x)(\phi^2+\psi^2)\,dx = p(b)\,[\psi(b)\phi'(b)-\phi(b)\psi'(b)] - p(a)\,[\psi(a)\phi'(a)-\phi(a)\psi'(a)]$$

$$=0 \qquad \text{By using eqs. (5)}$$

Since $r(x)\neq 0 \quad \forall x\in[a,b]$

So either $r(x)>0$ or $r(x)<0 \quad \forall x\in[a,b]$

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In either case $\int_a^b r(x)(\varphi_1^2 + \psi^2)\,dx \neq 0$

$\Rightarrow \lambda_2 = 0$

$\Rightarrow \lambda = \lambda_1 + i(0) = \lambda_1$

$\Rightarrow \lambda$ is real.

Square Integrable Functions:-

A function $f(x)$ is said to be square integrable with weight function $w(x) > 0$ over an interval $[a, b]$ if

$$\int_a^b w(x)\,|f(x)|^2\,dx < \infty$$

If $w(x) = 1$, then $\int_a^b |f(x)|^2\,dx < \infty$

In this case $f(x)$ is simply called ~~square~~ integrable.

Lagrang's Identity:-

Suppose $u(x)$ and $v(x)$ are two solutions of an S.L. equation, then the following identity must hold

$$uL(v) - vL(u) = \frac{d}{dx}\left[p(x)\{u(x)v'(x) - v(x)u'(x)\}\right]$$

which is called differential form of Lagrange's identity.

The integral form is given

$$\int_a^b [uL(v) - vL(u)]\,dx = p(x)\{u(x)v'(x) - v(x)u'(x)\}\Big|_a^b$$

Derivation:- Since

$$L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q r(x)$$

So

$$uL(v) - vL(u) = u\frac{d}{dx}\left\{p(x)\frac{dv}{dx}\right\} + qv ~~u(x)~~ v(x) - v\frac{d}{dx}\{p(x)$$

$- v(x)\, q(x)\, u(x)$

$= u \frac{d}{dx}\{p(x) v'(x)\} - v \frac{d}{dx}\{p(x) u'(x)\}$

$= u(x)\{p(x) v''(x) + p'(x) v'(x)\} - v(x)\{p(x) u''(x) + p'(x) u'\}$

$= u(x) p(x) v''(x) + u(x) p'(x) v'(x) - v(x) p(x) u''(x) - v(x) p'(x) u'(x)$

$= p(x)\{u(x) v''(x) - v(x) u''(x)\}$
$+ p'(x)\{u(x) v'(x) - v(x) u'(x)\}$

$= p(x)\{u(x) v'(x) - v(x) u'(x)\}' + p'(x)\{u(x) v'(x) - v(x) u'(x)\}$

$= \frac{d}{dx}\{p(x)(u(x) v'(x) - v(x) u'(x))\}$

$\Rightarrow u L(v) - v L(u) = \frac{d}{dx}\{p(x)(u(x) v'(x) - v(x) u'(x))\}$

Which is req. differential form of the Lagrange's identity. If we denote the Wronskian of the functions $u(x)$ and $v(x)$ by $W(u,v)(x)$. Then the above identity can also be written as

$u L(v) - v L(u) = \frac{d}{dx}\{p(x) W(u,v)(x)\}$

Question:-

Show by Lagrange's identity that ~~eigenfunction corresponding~~ to an eigenvalues of an S.L. System (regular or periodic) are real.

Sol: Let $u(x)$ be the eigenfunction corresponding to the eigenvalue λ.

Then $u(x) \neq 0 \quad \forall x \in [a,b]$

Now $L(u) + \lambda r(x) u = 0$

$\Rightarrow L(u) = -\lambda r(x) u \longrightarrow (1)$

Let λ be complex number. Thereby

① $L(\bar{u}) = -\bar{\lambda}\, r(x)\bar{u}$ $\because r$ is real.

Now from integral form of Lagrange's identity

$$\int_a^b [uL(v) - vL(u)]\,dx = p(x)\{uv' - vu'\}\Big|_a^b$$

Taking $v = \bar{u}$ then

$$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]\,dx = p(x)[u\bar{u}' - \bar{u}u']\Big|_a^b$$

$$= p(b)[u(b)\bar{u}'(b) - \bar{u}(b)u'(b)]$$

$$- p(a)[u(a)\bar{u}'(a) - \bar{u}(a)u'(a)] \rightarrow ①$$

Also that

$\alpha_1\bar{u}(a) + \alpha_2\bar{u}'(a) = 0 \Rightarrow \bar{u}'(a) = -\frac{\alpha_1}{\alpha_2}\bar{u}(a)$

$\beta_1\bar{u}(b) + \beta_2\bar{u}'(b) = 0 \Rightarrow \bar{u}'(b) = -\frac{\beta_1}{\beta_2}\bar{u}(b)$

$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \Rightarrow u'(a) = -\frac{\alpha_1}{\alpha_2}u(a)$

$\beta_1 u(b) + \beta_2 u'(b) = 0 \Rightarrow u'(b) = -\frac{\beta_1}{\beta_2}u(b)$

Using these values eq. ① becomes

$$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]\,dx = p(b)\left[u(b)\left(-\frac{\beta_1}{\beta_2}\bar{u}(b)\right) - \bar{u}(b)\left(-\frac{\beta_1}{\beta_2}u(b)\right)\right]$$

$$- p(a)\left[u(a)\left(-\frac{\alpha_1}{\alpha_2}\bar{u}(a)\right) - \bar{u}(a)\left(-\frac{\alpha_1}{\alpha_2}u(a)\right)\right]$$

$$= p(b)(0) - p(a)(0)$$

$$= 0$$

$$\Rightarrow \int_a^b [u(-\bar{\lambda}\, r(x)\bar{u}(x)) - \bar{u}(-\lambda\, r(x)u(x))]\,dx = 0$$

$$\Rightarrow (\lambda - \bar{\lambda})\int_a^b [-|u|^2 + |u|^2]\, r(x)\,dx = 0$$

$$\Rightarrow \int_a^b (\lambda u\bar{u} - \bar{\lambda} u\bar{u})\, r(x)\,dx = 0$$

$$\Rightarrow \int_a^b (\lambda - \bar{\lambda})\, u\bar{u}\, r(x)\,dx = 0 \Rightarrow (\lambda - \bar{\lambda})\int_a^b r(x)\, u\bar{u}\,dx = 0$$

Since $r(x) \neq 0$ & $u\bar{u} > 0$ so $\lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

Same is the case for Periodic S.L. System

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Orthogonality of Eigenfunctions:

Show by Lagrange's identity that eigen-functions of a regular or periodic S.L. System corresponding to distinct eigen-values are orthogonal w.r.t. the weight function $r(x)$.

Solution:-

Let u_m and u_n be the two non zero eigenfunctions of a (regular or periodic) S.L. System corresponding to two distinct eigenvalues λ_m and λ_n respectively. Then

$$L(u_m) = -\lambda_m r(x) u_m$$
$$L(u_n) = -\lambda_n r(x) u_n$$

Now by Lagrange's identity

$$\int_a^b \{u_m L(u_n) - u_n L(u_m)\} dx = p(x)\left[u_m u_n' - u_n u_m'\right]_a^b$$

$$= p(b)\left[u_m(b)\, u_n'(b) - u_n(b)\, u_m'(b)\right]$$
$$- p(a)\left[u_m(a) u_n'(a) - u_n(a) u_m'(a)\right]$$
$$= 0 \quad \text{(Both for regular \& Periodic S.L. Systems)}$$

$$\Rightarrow \int_a^b \left[u_m(-\lambda_n r(x) u_n) - u_n(-\lambda_m r(x) u_m)\right] dx = 0$$

$$\Rightarrow (\lambda_m - \lambda_n) \int_a^b r(x)\, u_m(x) u_n(x)\, dx = 0$$

As $\lambda_m - \lambda_n \neq 0$ for as $\lambda_m \neq \lambda_n$

So $$\int_a^b r(x)\, u_m(x) u_n(x)\, dx = 0$$

As Required.

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Simplicity of Eigenvalues.

The eigenvalues of a regular S. L. System are Simple.
Or
To each eigenvalue there corresponds only one linearly independent eigenfunction.
Or
If $u(x)$ and $v(x)$ are eigenfunctions corresponding to the same eigenvalue then they must differ by a multiplicative constant.

Proof:-
Let $u(x)$ and $v(x)$ be the two eigenfunctions corresponding to the same eigenvalue λ. Then

$$L(u) = -\lambda r(x) u$$

$$\& \; L(v) = -\lambda r(x) v$$

So $$uL(v) - vL(u) = u(-\lambda r(x) v) - v(-\lambda r(x) u)$$

$$= -u\lambda r(x) v + v\lambda r(x) u$$

$$\Rightarrow uL(v) - vL(u) = 0$$

$$\Rightarrow \frac{d}{dx}\{p(x)\, W(u,v)(x)\} = 0 \quad \forall x \in [a,b]$$

(By Lagrange's Identity)

$$\Rightarrow p(x)\, W(u,v)(x) = 0 \quad \forall x \in [a,b]$$

$$\Rightarrow W(u,v)(x) = 0 \quad \because p(x) \neq 0 \;\; \forall x \in [a,b]$$

$\Rightarrow$ $u(x)$ and $v(x)$ are linearly dependent.

Abel's Formula:-

If $u(x)$ and $v(x)$ are any two solutions of a regular or periodic S.L. system, then

$$p(x)\,W(u,v)(x) = \text{constant}, \quad x \in [a,b]$$

Proof: As we know that for a regular or periodic S.L. System

$u\mathcal{L}(v) - v\mathcal{L}(u) = 0$ for any pair of solutions $u(x)$ and $v(x)$.

So by differential form of Lagrange's identity $\frac{d}{dx}\{p(x)\,W(u,v)(x)\} = 0$

$\Rightarrow p(x)\,W(u,v)(x) = \text{constant} \quad \forall x \in [a,b]$

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Theorem:-

Any eigenvalue λ can be related to its eigenfunction $u(x)$ by the Rayleigh quotient

$$\lambda = \frac{-pu u'\Big|_a^b + \int_a^b (-pu'^2 - q u^2)\,dx}{\int_a^b r(x) u^2\,dx}$$

Proof:-

S. L. Eq. is

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda r(x)u(x) = 0$$

Multiplying above equation by u and integrating from a to b

$$\int_a^b \left\{u\frac{d}{dx}(pu') + q(x)u^2\right\}dx + \lambda\int_a^b r(x)u^2(x)\,dx = 0$$

$$\Rightarrow \lambda = -\frac{\int_a^b \{u(pu')' + q(x)u^2\}\,dx}{\int_a^b r(x)u^2\,dx}$$

$$= -\frac{\int_a^b u(pu')'\,dx + \int_a^b q u^2\,dx}{\int_a^b r(x)u^2\,dx}$$

$$= -\frac{u p u'\Big|_a^b - \int_a^b u'(pu')\,dx + \int_a^b q u^2\,dx}{\int_a^b r(x)u^2\,dx}$$

$$\lambda = \frac{-pu u'\Big|_a^b + \int_a^b (pu'^2 - q u^2)\,dx}{\int_a^b r(x)u^2\,dx}$$

It follows from this result that $\lambda \geq 0$ if $-puu'\Big|_a^b \geq 0$ and $q \leq 0$.

This result cannot be used to determine the eigenvalues; this is so because to evaluate λ we need to know the corresponding eigenfunction which in turn cannot be found without first knowing the eigenvalues.

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However interesting and important results can be obtained from the Rayleigh quotient without solving the diff. equation.

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Eigenfunction Expansion.

Consider the series

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x) \rightarrow ①$$

Where we have assumed that the series is uniformly convergent. Such a series is called generalized fourier series.

Where the coefficients a_n can be determined by using the orthogonality condition of the eigenfunctions as follows.

Xing both sides of ① by $r(x)u_m(x)$ and integrating from a to b and using the orthogonality conditions.

$$\int_a^b r(x) f(x) u_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_a^b r(x) u_m(x) u_n(x)\, dx$$

$$= a_m \int_a^b r(x)\, u_m(x)\, u_m(x)\, dx$$

$$\Rightarrow a_m = \frac{\int_a^b r(x) f(x) u_m(x)\, dx}{\int_a^b r(x) u_m^2(x)\, dx} \qquad m = 1, 2, 3, \ldots$$

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Self-adjoint operator:-

An operator A defined over a linear space of functions is said to be self-adjoint if

$$\langle u, Av \rangle = \langle Au, v \rangle$$

or equivalently

$$\int_a^b u(x)\,(Av(x))\,dx = \int_a^b (A\,u(x))\,v(x)\,dx$$

Here u and v are supposed to be real.

Theorem:-

Prove that S.L. Operator is self-adjoint

Proof:-

Any regular or periodic S.L. system is given by

$$L(u) + \lambda r(x) u = 0 \quad \text{where } L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$$

with suitable end point conditions.

Let u & v be the two eigen functions. Then by Lagrange's identity

$$\int_a^b [u L(v) - v L(u)]\,dx = p(x)\,[uv' - vu']\Big|_a^b$$

$$= 0 \quad (\text{for both regular \& Periodic S.L. Systems})$$

$$\Rightarrow \int_a^b u L(v)\,dx = \int_a^b v L(u)\,dx$$

$\Rightarrow$ L is self adjoint.

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EXAMPLES:-

Example 1:-

Verify that for the S.L. System
$u'' + \lambda u = 0$ $u'(0) = 0$ $u'(l) = 0$

(i) there are an infinite number of eigenvalues, with smallest but no largest.
(ii) The nth eigenfunction has exactly $n-1$ zeros.
(iii) The eigenfunctions are orthogonal and form a complete set.

Solution:-

A.E. is $D^2 + \lambda = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

$\Rightarrow u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$\Rightarrow u'(x) = -c_1 \sin\sqrt{\lambda}x \cdot \sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$u'(0) = 0$

$\Rightarrow \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x = 0$

$\Rightarrow \sqrt{\lambda}c_2 \cos\sqrt{\lambda}(0) = 0$

$\Rightarrow \sqrt{\lambda}c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$\Rightarrow u'(x) = -c_1 \sin\sqrt{\lambda}x \cdot \sqrt{\lambda}$

$\Rightarrow u'(l) = 0 \Rightarrow -c_1 \sin\sqrt{\lambda}l \cdot \sqrt{\lambda} = 0$

$\Rightarrow \sqrt{\lambda}c_1 \sin\sqrt{\lambda}l = 0$

Now $\sqrt{\lambda} \neq 0$ & $c_1 \neq 0$ $\because$ then the sol is trivial.

$\Rightarrow \sin\sqrt{\lambda}l = 0$

$\Rightarrow \sqrt{\lambda}l = n\pi$ $n = 0, \pm1, \pm2, \cdots$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{l}$

$\Rightarrow \lambda = \dfrac{n^2\pi^2}{l^2}$

$\Rightarrow \lambda_n = \dfrac{n^2\pi^2}{l^2}$ $n = 0, 1, 2, 3, \cdots$

Which are required eigenvalues and the corresponding eigenfunctions are

$$u_n = C_n \cos \frac{n\pi x}{l} \qquad n = 0, 1, 2, 3, \ldots$$

(i) It is clear that $\lambda_0 = 0$ is the smallest eigenvalue. The other eigenvalues are λ_n for $n = 1, 2, 3, \ldots$. Obviously there is no largest eigenvalue.

(ii) The eigenfunction corresponding to the nth eigenvalue is given by

$$u_{n-1}(x) = C_{n-1} \cos\left(\frac{(n-1)\pi x}{l}\right), \quad n \geq 1$$

or equivalently

$$u_n = C_n \cos\left(\frac{n\pi x}{l}\right) \quad n = 0, 1, 2, 3, \ldots$$

Now we prove that $u_n(x)$ has exactly $n-1$ zeros, $n = 0, 1, 2, \ldots$

When $n = 0$ then $u_0 = C_0$

which has no zeros.

When $n = 1$ then $u_1 = C_1 \cos\left(\frac{\pi x}{l}\right)$

$u_1 = 0 \Rightarrow \frac{\pi x}{l} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow x = \frac{l}{2}, \frac{3l}{2}, \frac{5l}{2}, \ldots$

But only $x = l/2 \in (0, l)$ other values lie outside the interval.

When $n = 2$, $u_2 = C_2 \cos\left(\frac{2\pi x}{l}\right)$

$u_2 = 0 \Rightarrow \cos\left(\frac{2\pi x}{l}\right) = 0$

$\Rightarrow \frac{2\pi x}{l} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow x = \frac{l}{4}, \frac{3l}{4}, \frac{5l}{4}, \ldots$

But only $x = \frac{l}{4}, \frac{3l}{4} \in (0, l)$ and $\frac{5l}{4} \notin (0, l)$

$\Rightarrow u_0$ has no zero in $(0, l)$

u_1 i.e. 2nd e.f. has one zero in $(0, l)$

u_2 i.e. 3rd e.f. has two zeros in $(0, l)$

⋮ ⋮

nth eigenfunction has $n-1$ zeros in $(0, l)$.

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E (iii) Since every function $f(x)$ defined in $[0,l]$ and satisfying some conditions can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

So eigenfunctions form a complete set and also as $\int_0^l r(x) u_m(x) u_n(x)\,dx = 0$ for $m \neq n$

implies $\{u_m(x)\}$ are orthogonal.

Q. Verify that $\lambda_n^2 = \left(\frac{n\pi}{\ln b}\right)^2$ and $u_n = C_n \sin(\lambda_n \ln x)$ are eigen values and e.fs. of $(xu')' + \lambda^2\left(\frac{1}{x}\right)u = 0$, $1 < x < b$ with $u(1) = 0$ and $u(b) = 0$.

Sol. Given $x^2 u'' + x u' + \lambda^2 u = 0$

Which is Cauchy's Euler's eq. (10.4 Method B.Sc.)

$$\Rightarrow u = A x^{i\lambda} + B x^{-i\lambda}$$

$$= ~~A\cos\lambda x~~ = A e^{i\lambda \ln x} + B e^{-i\lambda \ln x}$$

$$= C_1 \cos(\lambda \ln x) + C_2 \sin(\lambda \ln x) \quad ?$$

$u(1) = 0 \Rightarrow C_1 = 0$

$u(b) = 0 \Rightarrow C_2 \sin(\lambda \ln b) = 0$

$\Rightarrow \sin(\lambda \ln b) = 0 \quad \because C_2 \neq 0$

$\Rightarrow \lambda \ln b = n\pi \Rightarrow \lambda_n^2 = \left(\frac{n\pi}{\ln b}\right)^2 \quad n \in N$

and $u_n(x) = C_n \sin(\lambda_n \ln x)$

Q. Verify that $\lambda_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}$ and $u_n = \exp(-x/2)\sin\left(\frac{n\pi x}{a}\right)$ are resp. e.Vs. and e.fs. of $(e^x u')' + \lambda^2 e^x u = 0$ with $u(0) = 0 = u(a)$

Sol. $(e^x u')' + \lambda^2 e^x u = 0 \Rightarrow u'' + u' + \lambda^2 u = 0$

$$\Rightarrow u(x) = e^{-x/2}\left\{C_1 \cos\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{4\lambda^2-1}}{2}\right)x\right\}$$

$u(0) = 0 \Rightarrow C_1 = 0$

$u(a) = 0 \Rightarrow \lambda_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}$

& $u_n(x) = B_n \exp(-x/2)\sin\left(\frac{n\pi x}{a}\right)$

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Example 2:

Verify that for the S-L System $u'' + \lambda u = 0$ with $u'(0) = 0$, $u(l) = 0$

a) There are infinite number of eigenvalues with smallest but no largest.

b) The nth eigenfunction has exactly $n-1$ zeros.

c) The eigenfunctions are orthogonal and form a complete set.

Solution: Given $u'' + \lambda u = 0$

$\Rightarrow u = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$

$u' = -\sqrt{\lambda}\, C_1 \sin\sqrt{\lambda}\, x + \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda}\, x$

$u'(0) = 0 \Rightarrow \sqrt{\lambda}\, C_2 = 0$

$\Rightarrow C_2 = 0 \quad \because \sqrt{\lambda} \neq 0$

$\because$ if $\sqrt{\lambda} = 0$, then sol. is trivial

So $u = C_1 \cos\sqrt{\lambda}\, x$

$u(l) = 0 \Rightarrow C_1 \cos\sqrt{\lambda}\, l = 0$

$\Rightarrow \cos\sqrt{\lambda}\, l = 0 \quad \because$ for nontrivial sol. $C_1 \neq 0$

$\Rightarrow \sqrt{\lambda}\, l = (2n+1)\frac{\pi}{2} \qquad n \in \mathbb{Z}$

$\Rightarrow \lambda_n = (2n+1)^2 \frac{\pi^2}{4l^2} \qquad n = 0, 1, 2, 3, \ldots$

which are req. eigenvalues.

The corresponding eigenfunctions are

$u_n = C_n \cos\left(\frac{(2n+1)\pi}{2l}\right)x \qquad n = 0, 1, 2, 3, \ldots$

Now

a) Obviously there are infinite number of eigenvalues with $\lambda_0 = \frac{\pi^2}{4l^2}$ as the smallest eigenvalue but there is no largest eigenvalue.

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b) Eigenfunction corresponding to nth eigenvalue is given by

$$u_{n-1} = C_{n-1} \cos\left((2n-1)\frac{\pi}{2l}\right)x \qquad n=1,2,3,\ldots$$

Or equivalently

$$u_n = C_n \cos\left(\frac{(2n+1)\pi}{2l}\right)x \qquad n=0,1,2,\ldots$$

$n=0 \Rightarrow u_0 = C_0 \cos\left(\frac{\pi}{2l}\right)x$

$u_0 = 0 \Rightarrow C_0 \cos\left(\frac{\pi}{2l}\right)x = 0$

$\Rightarrow \cos\left(\frac{\pi}{2l}x\right) = 0 \qquad \because C_0 \neq 0$

$\Rightarrow \frac{\pi}{2l}x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\Rightarrow x = l, x = 3l$

As $l, 3l, \ldots \notin (0,l)$

$\Rightarrow u_0$ has no zero in $(0,l)$

$n=1 \Rightarrow u_1 = C_1 \cos\left(\frac{3\pi}{2l}x\right)$

$u_1 = 0 \Rightarrow \frac{3\pi}{2l}x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\Rightarrow x = l/3, l, \ldots$

As $l/3 \in (0,l)$ and $l, 3l, \ldots \notin (0,l)$

$\Rightarrow u_1$ has only 1 zero in $(0,l)$

$n=2 \Rightarrow u_2 = C_2 \cos\left(\frac{5\pi}{2l}\right)x$

$u_2 = 0 \quad \frac{5\pi}{2l}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\Rightarrow x = l/5, 3l/5, l, 7l/5, \ldots$

only $l/5, 3l/5 \in (0,l)$ and $l, 7l/5, \ldots \notin (0,l)$

$\Rightarrow u_2$ has exactly 2 zeros in $(0,l)$

$\Rightarrow u_0$ i.e. 1st e.f. has no zero in $(0,l)$

u_1 i.e. 2nd e.f. has 1 zero in $(0,l)$

u_2 i.e. 3rd e.f. has 2 zeros in $(0,l)$

$\vdots$

nth eigenfunction has exactly $n-1$ zeros in $(0,l)$

c) Eigen functions are orthogonal and form a complete set for half interval $[0,l]$. Every function

4410622

f(x) defined in this interval and satisfying some conditions, can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n U_n(x)$$

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Example 3:-

Show that the following B.Cs. yield self-adjoint problems

a) $u(0) = 0$ $u(L) = 0$

b) $u'(0) = 0$ $u(L) = 0$

c) $u(a) = u(b)$, $P(a)u'(a) = P(b)u'(b)$

Sol:-

a) Here $a = 0$ $b = L$

Let u & v be the two e.fs.

Then $u(0) = 0$ $u(L) = 0$

& $v(0) = 0$ $v(L) = 0$

Now $\int_0^L (u\mathcal{L}(v) - v\mathcal{L}(u))\,dx = P(x)[uv' - vu']\Big|_0^L$

$= P(L)[u(L)v'(L) - v(L)u'(L)]$

$- P(0)[u(0)v'(0) - v(0)u'(0)]$

$= P(L)[0 - 0] - P(0)[0 - 0] = 0$

$\Rightarrow \int_0^L u\mathcal{L}(v)\,dx = \int_0^L v\mathcal{L}(u)\,dx$

$\Rightarrow \mathcal{L}$ is self-adjoint.

b) $\int_0^L (u\mathcal{L}(v) - v\mathcal{L}(u))\,dx = P(L)[(0)v'(L) - (0)u'(L)]$

$- P(0)[u(0)(0) - v(0)(0)]$

$= 0$

$\Rightarrow \int_0^L u\mathcal{L}(v)\,dx = \int_0^L v\mathcal{L}(u)\,dx$

$\Rightarrow \mathcal{L}$ is self adjoint

c) $\int_a^b (u\mathcal{L}(v) - v\mathcal{L}(u))\,dx = P(b)[u(b)v'(b) - u'(b)v(b)]$

$- P(a)[u(a)v'(a) - u'(a).v(a)]$

$= P(b)u(b)v'(b) - P(b)u'(b)v(b)$

$- P(a)u(a)v'(a) + P(a)u'(a)v(a)$

$= \sout{u(a)P(a)v'(a)} - P(a)u'(a)v(a) - P(a)u(a)v'(a) + P(a)u'(a)...$

$= 0$ $\mathcal{L}$ is self-adjoint

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Example 3.1 = 4

Using the Rayleigh quotient, discuss the sign of the eigenvalues of the S.L. System

$u'' + \lambda u = 0 \quad , \; u(0) = 0 \quad u(l) = 0$

$p(x) = 1 \quad q(x) = 0 \quad r(x) = 1$

Solution:

Rayleigh quotient is given by

$$\lambda = \frac{-u(1)u'\big|_0^l + \int_0^l \left((1)u'^2 - (0)u^2\right)du}{\int_0^l (1)u^2 dx}$$

$$= \frac{-\left(u(l)u'(l) - u(0)u'(0)\right) + \int_0^l u'^2 dx}{\int_0^l u^2 dx}$$

$$\lambda = \frac{\int_0^l u'^2 dx}{\int_0^l u^2 dx} \rightarrow (1)$$

Since u is unknown so λ cannot be determined from (1). Now as u is an e.f. So $u \neq 0 \Rightarrow u^2 > 0$ & $u'^2 \geq 0$

$\Rightarrow \lambda \geq 0$

$\Rightarrow \lambda$ cannot have the negative sign.

Now $\lambda = 0 \Leftrightarrow u' = 0$

$\Rightarrow u$ is constant $\Rightarrow u = c$

Now $u(0) = 0 \Rightarrow c = 0$ & $u(l) = 0 \Rightarrow c = 0$

$\Rightarrow c = 0 \Rightarrow u = 0$ which is trivial sol.

So $\lambda \neq 0 \Rightarrow \lambda > 0$

Hence all the eigen values are +ve.

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GREEN'S FUNCTIONS & ASSOCIATED BOUNDARY VALUE PROBLEMS

THE DIRAC DELTA FUNCTION:

KRONECKER DELTA FUNCTION

δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

It is a tensor of rank 2 and it has the following well known property

$$\sum_i \delta_{ij} a_i = a_j$$

In δ_{ij}, the indices i and j have the integral values as 1,2,3.

Dirac Delta Function is actually the generalization of the kronecker delta function δ_{ij}.

It is denoted and defined by

$$\delta(x-t) = \begin{cases} 0 & x \neq t \\ \infty & x = t \end{cases}$$

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Contrary to δ_{ij}, in $\delta(x-t)$, the values of x and t are real continuous.

Properties.

It has the following properties

(i) $\int_{-\infty}^{\infty} \delta(x-t)\,dx = 1$

(ii) $\int_{-\infty}^{\infty} \delta(x-t) f(x)\,dx = f(t)$

Motivation for Green's Function.

We have to solve the problem associated with non-homogeneous diff. eq.

$$L\{u(x)\} + \lambda\, r(x)\, u(x) = f(x) \longrightarrow (1)$$

where $L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

and u satisfies some suitable end-point conditions.

The sol. of non-homogeneous diff. eq. subject to B.Cs. is closely related to the existance of Green's Function associated with homogeneous eq. $L\{u\} + \lambda\, r(x)\, u = 0$

If a function $G(x,t,\lambda)$ which does not depend on the source function $f(x)$ exists then the sol. of (1) can be written as

$$u(x) = \int_a^b G(x,t,\lambda)\, f(t)\,dt$$

In this case $G(x,t,\lambda)$ is called Green's Function and satisfies the eq.

$$L\{G\} + \lambda\, r(x)\, G = \delta(x-t)$$

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Green's Function Associated With Regular S.L. System

Let

$L(u) + \lambda r(x) u = 0 \longrightarrow (1)$ be the S.L. equation with endpoint conditions

$\alpha u(a) + \alpha' u'(a) = 0 \longrightarrow (2)$

and $\beta u(b) + \beta' u'(b) = 0 \longrightarrow (3)$

Eqs. (2) and (3) can also be written as $B_1(u) = 0$ and $B_2(u) = 0$

Under the assumption that $\lambda = 0$ is not an eigenvalue of this system (Regular S.L. System) i.e. it gives trivial solution, and there associates a function with it, called Green's Function which has the following properties:

1) Green's Function $G(x,t)$ considered as a function of x satisfies the differential equation $L\{G(x,t)\} = 0$ in each of the subinterval $[a,t)$ and $(t,b]$

2) $G(x,t)$ is continuous in $[a,b]$

3) $G(x,t)$ as a function of x satisfies the end-point conditions $B_1(G) = 0$ and $B_2(G) = 0$

4) $\frac{dG}{dx}$ is discontinuous as $x \to t$ and moreover

$$\lim_{x \to t-0} G'(x,t) - \lim_{x \to t+0} G'(x,t) = \frac{1}{P(t)}$$

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EXAMPLES:-

Example 1:-

Construct Green's function associated with the problem

$u'' + \lambda u = 0$

with $u(0) = 0$ and $u(1) = 0$

Sol:-

1) First : we check whether $\lambda = 0$ is eigenvalue or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u' = A$

$\Rightarrow u = Ax + B$

$u(0) = 0 \Rightarrow A(0) + B = 0 \Rightarrow \boxed{B = 0}$

$u(1) = 0 \Rightarrow A(1) + B = 0 \Rightarrow A + 0 = 0$

$\Rightarrow \boxed{A = 0}$

$\Rightarrow u = 0$

$\Rightarrow \lambda = 0$ is not the eigenvalue

2) $G(x,t)$ satisfies the diff. eq.

$\Rightarrow G''(x,t) = 0$ in each of the subinterval $[0,t), (t,1]$

Hence

$$G(x,t) = \begin{cases} Ax + B & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

3) Since $G(x,t)$ is continuous in $[0,1]$

So is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$\Rightarrow A(t-0) + B = A'(t+0) + B'$

$\Rightarrow At + B = A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

4410622

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4) $G(x,t)$ satisfies the end point conditions $G(0,t)=0$ and $G(1,t)=0$

$\Rightarrow A(0)+(A'-A)t+B'=0$

$\Rightarrow (A'-A)t+B'=0 \Rightarrow B'=(A-A')t$

and $A'(1)+B'=0 \Rightarrow B'=-A'$

So $-A'=(A-A')t$

$\Rightarrow -A'+A't=At \Rightarrow A=\frac{(t-1)}{t}A'$

Hence $G(x,t)=\begin{cases}\frac{(t-1)}{t}A'x+0 & 0\le x<t\\ A'x-A' & t<x\le 1\end{cases}$

5) $G(x,t)$ satisfies the discontinuity condition

$G'(t-0,t)-G'(t+0,t)=\frac{1}{p(t)}$

$(t-1)\frac{A'}{t}-A'=1$

$\Rightarrow$ ~~$A't-A'$~~ $\qquad A'-\frac{A'}{t}-A'=1$

$\Rightarrow$ ~~$A'=\frac{t}{t-1}$~~ $\qquad \Rightarrow A'=-t$

So $G(x,t)=\begin{cases}\left(\frac{t-1}{t}\right)(-t)x & 0\le x<t\\ (x-1)(-t) & t<x\le 1\end{cases}$

$\Rightarrow G(x,t)=\begin{cases}x(1-t) & 0\le x<t\\ t(1-x) & t<x\le 1\end{cases}$

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Example 2:-

Construct Green's function associated with the problem

$$xu'' + u' + \lambda r(x) u = 0$$

$u(0)$ is finite and $u(1) = 0$

Sol:-

1) First we check whether $\lambda = 0$ is eigenvalue or not.

$\lambda = 0 \Rightarrow xu'' + u' = 0$

$$\frac{d}{dx}(xu') = 0$$

$\Rightarrow xu' = A \quad \Rightarrow u' = \frac{A}{x}$

$\Rightarrow u(x) = A \ln x + B$

$u(0)$ is finite $\Rightarrow A = 0$

$\Rightarrow u(x) = B \qquad u(1) = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not the eigenvalue of the system.

2) $G(x,t)$ satisfies the diff. eq.

$x G''(x,t) + G'(x,t) = 0$ in each of the subinterval $[0,t)$ and $(t,1]$

So
$$G(x,t) = \begin{cases} A \ln x + B & 0 < x < t \\ A' \ln x + B' & t < x \le 1 \end{cases}$$

3) Since $G(x,t)$ is continuous in $[0,1]$ so is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$\Rightarrow A \ln t + B = A' \ln t + B'$

$\Rightarrow -B' = (A' - A) \ln t - B$

$\Rightarrow B' = (A - A') \ln t + B$

$$\Rightarrow G(x,t) = \begin{cases} A \ln x + B & 0 \le x < t \\ A' \ln x + (A - A') \ln t + B & t < x \le 1 \end{cases}$$

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4) $G(x,t)$ satisfies the end point conditions

$G(0,t)$ is finite $\Rightarrow$ $\boxed{A=0}$

$G(1,t)=0$

$\Rightarrow A'\ln(1)+(A-A')\ln t+B=0$

$\Rightarrow B=(A'-A)\ln t$

$$G(x,t)=\begin{cases}(A'-0)\ln t & 0\le x<t\\ A'\ln x & t<x<1\end{cases}$$

5) By the discontinuity condition

$$G'(t-0,t)-G'(t+0,t)=\frac{1}{p(t)}$$

$$0-A'/t=\frac{1}{t}\Rightarrow A'=-1$$

$$\Rightarrow G(x,t)=\begin{cases}-\ln t & 0\le x<t\\ -\ln x & t<x\le 1\end{cases}$$

Example 3:

Construct Green's function associated with the problem

$$xu''+u'-\frac{n^2}{x}u+\lambda r(x)u=0$$

with $u(0)$ is finite and $u(1)=0$

Sol: Given eq: can be written as

$$\frac{d}{dx}(xu')-\frac{n^2}{x}u+\lambda r(x)u=0$$

Here

$$p(x)=x \qquad q(x)=\frac{-n^2}{x}$$

$$a=0 \quad \& \quad b=1$$

1) First we check whether $\lambda=0$ is eigenvalue or not.

$$\lambda=0\Rightarrow\frac{d}{dx}(xu')-\frac{n^2}{x}u=0$$

$$\Rightarrow\frac{d}{dx}(xu')=\frac{n^2}{x}u$$

~~integrating both sides.~~

~~$xu' = n^2 \ln x$~~

$\Rightarrow xu'' + u' - \frac{n^2}{x}u = 0$

$\Rightarrow x^2u'' + xu' - n^2u = 0.$

Which is Cauchy's Euler Equation

[Exercise 10.4 Mathematical Methods B.Sc.]

$u = Ax^n + Bx^{-n}$

$u(0)$ is finite $\Rightarrow B = 0$

$u(x) = Ax^n \qquad u(1) = 0 \Rightarrow A = 0$

$\Rightarrow \lambda = 0$ is not the eigenvalue

2) $G(x,t)$ satisfies differential eq. in each of the subinterval $[0,t)$ and $(t,1]$

$$\Rightarrow G(x,t) = \begin{cases} Ax^n + Bx^{-n} & 0 \le x < t \\ A'x^n + B'x^{-n}. & t < x \le 1 \end{cases}$$

3) As $G(x,t)$ is continuous on $[0,1]$

So is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$At^n + Bt^{-n} = A't^n + B't^{-n}$

$At^{2n} + B = A't^{2n} + B'$

$\Rightarrow B = (A' - A)t^{2n} + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax^n + [(A'-A)t^{2n} + B']x^{-n} & 0 \le x < t \\ A'x^n + B'x^{-n} & t < x \le 1 \end{cases}$$

4) $G(x,t)$ satisfies the end point conditions

$G(0,t)$ is finite and $G(1,t) = 0$

$G(0,t)$ is finite $\Rightarrow (A' - A)t^{2n} + B' = 0$

$G(1,t) = 0 \Rightarrow A' + B' = 0 \Rightarrow B' = -A'$

$$\Rightarrow G(x,t) = \begin{cases} Ax^n & 0 \le x < t \\ A'x^n - A'x^{-n} & t < x \le 1 \end{cases}$$

and $(A' - A)t^{2n} - A' = 0$

$\Rightarrow A't^{2n} - A' = At^{2n} \Rightarrow A = A'(t^{2n} - 1)/t^{2n}$

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So $G(x,t) = \begin{cases} A'\left(1-\frac{1}{t^{2n}}\right)x^n & 0 \le x < t \\ A'(x^n - x^{-n}) & t < x \le 1 \end{cases}$

5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$A'(1-t^{-2n})\,n t^{n-1} - A'(n t^{n-1} + n t^{-n-1}) = \frac{1}{t}$$

$$\Rightarrow A'(1-t^{-2n})\,n t^{n} - nA'(t^{n} + t^{-n}) = 1$$

$$\Rightarrow \cancel{nA't^{n}} - nA't^{-n} - \cancel{nA't^{n}} - nA't^{-n} = 1$$

$$-2nA't^{-n} = 1$$

$$\Rightarrow A' = \frac{t^n}{-2n} = -\frac{1}{2n}t^n$$

$$\Rightarrow G(x,t) = \begin{cases} -\frac{1}{2n}(t^n - t^{-n})x^n & 0 \le x < t \\ -\frac{1}{2n}t^n(x^n - x^{-n}) & t < x \le 1 \end{cases}$$

Example 4:-

Construct the Green's function for $\frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u + \lambda r(x)u = 0$ with $u(\pm 1)$ are finite.

Sol:-

Here $p(x) = 1-x^2$, $q(x) = -\frac{h^2}{1-x^2}$

$a = -1$ $b = 1$

1) First we check whether $\lambda = 0$ is eigenvalue or not

$\lambda = 0 \Rightarrow \frac{d}{dx}\{(1-x^2)u'\} - \frac{h^2}{1-x^2}u = 0$

$$\Rightarrow (1-x^2)u'' - 2xu' - \frac{h^2}{1-x^2}u = 0 \rightarrow (1)$$

Put $t = \ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$

$$\frac{dt}{dx} = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$$

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$$\frac{du}{dx}=\frac{du}{dt}\cdot\frac{dt}{dx}=\frac{2}{1-x^2}\frac{du}{dt}$$

$$\frac{d^2u}{dx^2}=\frac{d}{dx}\left(\frac{2}{1-x^2}\frac{du}{dt}\right)$$

$$=-2(1-x^2)^{-2}(-2x)\frac{du}{dt}+\frac{2}{1-x^2}\frac{d}{dt}\left(\frac{du}{dt}\right)\cdot\frac{dt}{dx}$$

$$=\frac{4x}{(1-x^2)^2}\frac{du}{dt}+\frac{2}{1-x^2}\cdot\frac{2}{1-x^2}\frac{d^2u}{dt^2}$$

$$=\frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2}+x\frac{du}{dt}\right]$$

Putting these values in eq. ① we get

$$\cancel{(1-x^2)}\cdot\frac{4}{(1-x^2)^2}\left\{\frac{d^2u}{dt^2}+x\frac{du}{dt}\right\}-2x\cdot\frac{2}{1-x^2}\frac{du}{dt}-\frac{h^2}{1-x^2}u=0$$

$$4\left(\frac{d^2u}{dt^2}+x\frac{du}{dt}\right)-4x\frac{du}{dt}-\frac{h^2}{1}u=0$$

$$\Rightarrow 4\frac{d^2u}{dt^2}-h^2u=0$$

$$\Rightarrow \frac{d^2u}{dt^2}-\frac{h^2}{4}u=0$$

$$\Rightarrow u=Ae^{ht/2}+Be^{-ht/2}$$

$$\Rightarrow u=A\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h}+B\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h}$$

$u(1)$ is finite $\Rightarrow A=0$

$u(-1)$ is finite $\Rightarrow B=0$

$\Rightarrow u=0$

$\Rightarrow \lambda=0$ is not the eigenvalue.

2) $G(x,t)$ satisfies the given diff. eq.
$(1-x^2)G''-2xG'-\frac{h^2}{1-x^2}G=0$ in each of the subinterval $[-1,t)$ and $(t,1]$

$$\Rightarrow G(x,t)=\begin{cases}A\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h}+B\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h} & -1<x<t\\ A'\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h}+B'\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h} & t<x\le 1\end{cases}$$

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3) $G(x,t)$ satisfies the end point conditions.

$G(-1,t)$ is finite $\Rightarrow B = 0$

$G(1,t)$ is finite $\Rightarrow A' = 0$

$$\Rightarrow G(x,t) = \begin{cases} A\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h} & -1 \le x < t \\ B'\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h} & t < x \le 1 \end{cases}$$

(4) $G(x,t)$ is continuous at $x=t$

$$\Rightarrow G(t-0,t) = G(t+0,t)$$

$$\Rightarrow A\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h} = B'\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h} = C$$

$$\Rightarrow A = \left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h} C \qquad B' = \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h} C$$

$$\Rightarrow G(x,t) = \begin{cases} \left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h} C & -1 \le x < t \\ \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h} C & t < x \le 1 \end{cases}$$

(5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h} \cdot \frac{1}{2}h\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h-1} \cdot C \cdot \left(\frac{2}{(1-t)^2}\right) - \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h}\left(-\frac{1}{2}h\right)\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h-1}\left(\frac{2}{(1-t)^2}\right) \cdot C =$$

$$\frac{hC}{1-t^2} + \frac{hC}{1-t^2} = \frac{1}{1-t^2} \Rightarrow C = \frac{1}{2h}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{1}{2h}\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}h}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}h} & -1 \le x < t \\ \frac{1}{2h}\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}h}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}h} & t < x \le 1 \end{cases}$$

Example 5:-

Construct Green's Function associated with the problem

$$u'' + \lambda u = 0$$

with $u(0) + u'(1) = 0$, $u(1) + 2u'(0) = 0$.

Solution:-

(1) First we check whether $\lambda = 0$ is eigen value or not.

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$

$u'(x) = A$

$u(0) + u'(1) = 0 \Rightarrow B + A = 0 \rightarrow$ (i)

$u(1) + 2u'(0) = 0 \Rightarrow A + B + 2A = 0 \Rightarrow 3A + B = 0 \rightarrow$ (ii)

From (i) & (ii) $A = 0$ & $B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not the eigen value.

(2) $G(x,t)$ satisfied the diff. eq.
$G''(x,t) = 0$ in each of the subinterval $[0,t)$ and $(t,1]$

$$\Rightarrow G(x,t) = \begin{cases} Ax + B & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

(3) $G(x,t)$ is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$At + B = A't + B' \Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

(4) $G(x,t)$ satisfies the end point conditions. So

$(A' - A)t + B' + A' = 0$

and $A' + B' + 2A = 0 \rightarrow *$

$\Rightarrow (A' - A)t - 2A = 0$

$\Rightarrow A' = (2A + At)/t = A\left(\frac{t+2}{t}\right)$

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and $A\left(\frac{t+2}{t}\right) + B' + 2A = 0$ by x

$$\frac{2}{t}A + B' + 3A = 0$$

$$B' = -\left(3+\frac{2}{t}\right)A$$

$$\Rightarrow G(x,t) = \begin{cases} Ax - A\left(\frac{t+2}{t}\right) & 0 \le x < t \\ A\left(\frac{t+2}{t}\right)x - A\left(\frac{3t+2}{t}\right) & t < x \le 1 \end{cases}$$

5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$A - A\left(1+\frac{2}{t}\right) = \frac{1}{1} \Rightarrow -\frac{2A}{t} = 1$$

$$\Rightarrow A = -\frac{1}{2}t$$

$$\Rightarrow G(x,t) = \begin{cases} -\frac{1}{2}tx + \frac{1}{2}t\left(\frac{t+2}{t}\right) & 0 \le x < t \\ -\frac{1}{2}t\left(\frac{t+2}{t}\right)x + \frac{1}{2}t\left(\frac{3t+2}{t}\right) & t < x \le 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{2}tx + \frac{1}{2}t + 1 & 0 \le x < t \\ \left(-\frac{1}{2}t - 1\right)x + \frac{3}{2}t + 1 & t < x \le 1 \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{1}{2}(2 + t - tx) & 0 \le x < t \\ \frac{1}{2}(2 + 3t - tx - 2x) & t < x \le 1 \end{cases}$$

Example 6:

Construct Green's Function associated with the system

$u'' + \lambda u = 0$

with $u'(0) = 0$, $u(1) = 0$

Sol:

Here $p(x) = 1$ $a = 0$ $b = 1$

1) First we check whether $\lambda = 0$ is eigenvalue or not

put $\lambda = 0$ $\Rightarrow u'' = 0$ $\Rightarrow u = Ax + B$

$u(1) = 0 \Rightarrow A + B = 0$

$u'(0) = 0 \Rightarrow A = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not the eigenvalue

2) $G(x,t)$ satisfies the diff. eq.

$G'' = 0$ in $[0,t)$ and $(t,1]$

$$\Rightarrow G(x,t) = \begin{cases} Ax + B & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

(3) As $G(x,t)$ is continuous at $x = t$

so $G(t-0,t) = G(t+0,t)$

$\Rightarrow At + B = A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

(4) $G(x,t)$ satisfies the endpoint conditions

$G(1,t) = 0 \Rightarrow A' + B' = 0 \rightarrow (1)$

$G'(0,t) = 0 \Rightarrow A = 0$

By (1) $B' = -A'$

$$\text{So } G(x,t) = \begin{cases} A't - A' & 0 \le x < t \\ A'x - A' & t < x \le 1 \end{cases}$$

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(5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$0 - A' = \frac{1}{1} \Rightarrow A' = -1$$

So
$$G(x,t) = \begin{cases} 1-t & 0 < x < t \\ 1-x & t < x \leq 1 \end{cases}$$

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MODIFIED GREEN'S FUNCTION

We know that if $\lambda = 0$ is not the eigenvalue of the S.L. System
$L(u) + \lambda r(x) u = 0$
with $B_1(u) = 0$, $B_2(u) = 0$
then we can associate a function with it having certain properties and is called Green's Function.
If $\lambda = 0$ is eigenvalue of the S.L. System defined above then the associated Green's Function is called
Modefied Green's Function.
It is denoted by $G_M(x,t)$ and can be constructed as follows
1) Let $u_0(x)$ be the normalized eigenfunction corresponding to $\lambda = 0$
This means that
$$\langle u_0, u_0 \rangle = \int_a^b u_0(x) u_0(x)\, dx = 1$$
2) $G_M(x,t)$ satisfies the eq.
$$L[G_M(x,t)] = u_0(x)\, u_0(t)$$
in each of the subinterval $[a,t)$ and $(t,b]$
3) $G_M(x,t)$ satisfies the end point conditions
$$B_1(G_M) = 0 \quad , \quad B_2(G_M) = 0$$
4) $G_M(x,t)$ is continuous everywhere in $[a,b]$; and is continuous particularly at $x = t$.

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5.) $G_M(x,t)$ satisfies the ~~end - poi~~ discontinuity condition

$$G'_M(t-0,t) - G'_M(t+0,t) = \frac{1}{p(t)}$$

(6) $G_M(x,t)$ satisfies the orthogonali condition

$$\int_a^b G_M(x,t)\, u_0(x)\, dx = 0$$

The above conditions uniquely determine the modified Green's Function.

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L = Double Derivatives

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EXAMPLES:-

Example 1:-

Construct Green's Function Associated with the system

$$u'' + \lambda\, r(x)\, u = 0$$

with $u'(0) = 0 = u'(1)$.

Solution:-

Here $p(x) = 1$

1) First we check whether $\lambda = 0$ is eigenvalue or not.

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$

$u'(x) = A \qquad u'(0) = 0 \Rightarrow A = 0$

$u'(1) = 0 \Rightarrow A = 0$

$\Rightarrow u = B \neq 0$

$\Rightarrow \lambda = 0$ is eigenvalue of the system. So associated Green's Function is the Modified Green's Function.

2) Let $u_0(x)$ be the normalized eigenfunction

So $\int_0^1 B \cdot B\, dx = 1 \Rightarrow B^2 x\big|_0^1 = 1 \Rightarrow B^2 = 1$

$\Rightarrow B = 1$

So $u_0(x) = 1$.

3) $G_M(x,t)$ satisfies the Diff. Eq.

$G''_M(x,t) = u_0(x)\, u_0(t)$ in $[0,t)$ and $(t,1]$

$G''_M(x,t) = (1)(1) = 1$

$G'_M(x,t) = x + A$

$G_M(x,t) = \frac{1}{2}x^2 + Ax + B$

So $G_M(x,t) = \begin{cases} \frac{1}{2}x^2 + Ax + B & 0 \le x < t \\ \frac{1}{2}x^2 + A'x + B' & t < x \le 1 \end{cases}$

④ $G_M(x,t)$ is cont. at $x=t$

$\Rightarrow G_M(t-0,t) = G_M(t+0,t)$

$\Rightarrow \frac{t^2}{2} + At + B = \frac{t^2}{2} + A't + B'$

$\Rightarrow B = (A'-A)t + B'$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + (A'-A)t + B' & 0 \le x < t \\ \frac{x^2}{2} + A'x + B' & t < x \le 1 \end{cases}$$

⑤ $G_M(x,t)$ satisfies the end-point conditions.

$G_M'(0,t) = 0 \Rightarrow 0 + A = 0 \Rightarrow A = 0$

$G_M'(1,t) = 0 \Rightarrow 1 + A' = 0 \Rightarrow A' = -1$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} - t + B' & 0 \le x < t \\ \frac{x^2}{2} - x + B' & t < x \le 1 \end{cases}$$

⑥ The discontinuity condition doesnot help in determining the unknown constants.

⑦ By the orthogonality condition

$$\int_0^1 G_M(x,t)\, u_0(x)\, dx = 0$$

$$\Rightarrow \int_0^t \left(\frac{x^2}{2} - t + B'\right) dx + \int_t^1 \left(\frac{x^2}{2} - x + B'\right) dx = 0$$

$$\frac{x^3}{6} - tx + B'x \Big|_0^t + \frac{x^3}{6} - \frac{x^2}{2} + B'x \Big|_t^1 = 0$$

$$\frac{t^3}{6} - t^2 + B't + \frac{1}{6} - \frac{1}{2} + B' - \frac{t^3}{6} + \frac{t^2}{2} - B't = 0$$

$$-\frac{t^2}{2} - \frac{1}{3} + B' = 0 \Rightarrow B' = \frac{t^2}{2} + \frac{1}{3}$$

$$\therefore G_M(x,t) = \begin{cases} \frac{x^2}{2} - t + \frac{t^2}{2} + \frac{1}{3} & 0 \le x < t \\ \frac{x^2}{2} - x + \frac{t^2}{2} + \frac{1}{3} & t < x \le 1 \end{cases}$$

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Example 2:-

Construct G.F. associated with the problem

$$u'' + \lambda u = 0$$

$u(0) = u(1)$ and $u'(0) = u'(1)$

Sol:- Here $a = 0$ $b = 1$ $P(x) = 1$

1) First we check whether $\lambda = 0$ is e.v. or not:

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u(x) = Ax + B$

$u(0) = u(1) \Rightarrow A(0) + B = A(1) + B \Rightarrow A = 0$

$u'(0) = u'(1) \Rightarrow A = A = 0 \Rightarrow A = 0$

$\Rightarrow u(x) = B$

$\Rightarrow \lambda = 0$ is eigen value.

So corresponding G.F. is Modified G.F.

② Let $u_0(x)$ be the normalized e.f.

Then $\int_0^1 B \cdot B \cdot dx = 1 \Rightarrow u_0(x) = 1$

③ $G_M(x,t)$ satisfies $G''_M(x,t) = u_0(x)\, u_0(t) = 1$

$\Rightarrow G_M(x,t) = \frac{x^2}{2} + Ax + B$

So for each subinterval $[0,t)$ and $(t,1]$ we can write

$$G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & 0 \le x < t \\ \frac{x^2}{2} + A'x + B' & t < x \le 1 \end{cases}$$

④ As $G_M(x,t)$ is continuous at $x = t$

So $G_M(t-0,t) = G_M(t+0,t)$

$\frac{t^2}{2} + At + B = \frac{t^2}{2} + A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + (A' - A)t + B' \\ \frac{x^2}{2} + A'x + B' \end{cases}$$

4410622

(5) $G_M(x,t)$ satisfies the end-point conditions

$G_M'(0,t) = G_M'(1,t)$

$A = 1 + A'$

and $G_M(0,t) = G_M(1,t)$

$\Rightarrow (A'-A)t + B' = \frac{1}{2} + A' + B'$

$\Rightarrow (A' - 1 - A')t = \frac{1}{2} + A'$

$\Rightarrow A' = -t - \frac{1}{2}$

So $A = 1 + A' = 1 - t - \frac{1}{2} = -t + \frac{1}{2}$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + (1-2t)x + (\frac{1}{2} - 2t)t + B' & 0 < x < t \\ \frac{x^2}{2} - 2xt + B' & t < x < 1 \end{cases}$$

(6) The discontinuity does not help us to determine B'

(7) By the orthogonality condition

$$\int_0^1 G_M(x,t)\, u_0(x)\, dx = 0$$

$$\Rightarrow \int_0^t \left[\frac{x^2}{2} + (1-2t)x + (\tfrac{1}{2} - 2t)t + B'\right] dx + \int_t^1 \left(\frac{x^2}{2} - 2xt + B'\right) dx = 0$$

$$\frac{x^3}{6} + (1-2t)\frac{x^2}{2} + (\tfrac{1}{2} - 2t)tx + B'x \Big|_0^t + \frac{x^3}{6} - x^2t + B'x \Big|_t^1 = 0$$

$$\frac{t^3}{6} + (1-2t)\frac{t^2}{2} + (\tfrac{1}{2} - 2t)t^2 + B't - 0 + \frac{1}{6} - t + B' - \frac{t^3}{6} + t^3 - B't = 0$$

$$\Rightarrow \frac{t^2}{2} - t^3 + \frac{1}{2}t^2 - 2t^3 + \frac{1}{6} - t + B' + t^3 = 0$$

$$\Rightarrow B' = 2t^3 - t^2 + t - \frac{1}{6}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + x - 2xt + \frac{1}{2}t - 3t^2 + 2t^3 - \frac{1}{6} & 0 < x < t \\ \frac{x^2}{2} - 2xt + 2t^3 - t^2 + t - \frac{1}{6} & t < x < 1 \end{cases}$$

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$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + (\frac{1}{2} - t)x - t + B' & 0 \le x < t \\ \frac{x^2}{2} - (t + \frac{1}{2})x + B' & t < x \le 1 \end{cases}$$

⑥ The discontinuity doesnot help us to determine B'

⑦ By the orthogonality condition

$$\int_0^1 G_M(x,t)\, u_0(x)\, dx = 0$$

$$\Rightarrow \int_0^t \left(\frac{x^2}{2} + (\tfrac{1}{2} - t)x - t + B'\right) dx + \int_t^1 \left[\frac{x^2}{2} - (t + \tfrac{1}{2})x + B'\right] dx = 0$$

$$\frac{t^3}{6} + (\tfrac{1}{2} - t)\cdot\frac{t^2}{2} - t^2 + B't + \frac{1}{6} - (t + \tfrac{1}{2})\cdot\frac{1}{2} + B'$$
$$- \frac{t^3}{6} + (t + \tfrac{1}{2})\cdot\frac{t^2}{2} - B't = 0$$

$$\frac{t^2}{4} - \frac{t^3}{2} - t^2 + \frac{1}{6} - \frac{t}{2} - \frac{1}{4} + B' + \frac{t^3}{2} + \frac{t^2}{4} = 0$$

$$-\frac{1}{2}t^2 - \frac{1}{12} - \frac{t}{2} + B' = 0$$

$$\Rightarrow B' = \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{12}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + \frac{x}{2} - tx - \frac{t}{2} + \frac{1}{2}t^2 + \frac{1}{12} & 0 \le x < t \\ \frac{x^2}{2} - tx - \frac{x}{2} + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{12} & t < x \le 1 \end{cases}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{1}{2}x^2 - tx + \frac{1}{2}x + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{12} & 0 \le x < t \\ \frac{1}{2}x^2 - tx - \frac{1}{2}x + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{12} & t < x \le 1 \end{cases}$$

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Example 3:

Construct Green's Function associated with the Problem

$$u'' + \lambda u = 0$$

with $u(-1) = u(1)$ and $u'(-1) = u'(1)$

Sol:-

1) First we check whether $\lambda = 0$ is eigenvalue or not.

$\lambda = 0 \Rightarrow u = Ax + B$

$u(-1) = u(1) \Rightarrow -A + B = A + B \Rightarrow A = 0$

$u'(-1) = u'(1) \Rightarrow A = A \Rightarrow A = 0$

$u \neq 0 \Rightarrow \lambda = 0$ is eigenvalue.

$\Rightarrow$ Associated G.F. is M.G.F.

(2) Let $u_0(x)$ be the normalized e.f.

Then $\int_{-1}^{1} u_0(x) u_0(x)\, dx = 1$

$\Rightarrow B^2 x \Big|_{-1}^{1} = 1 \Rightarrow B = 1/\sqrt{2}$

$\Rightarrow u_0(x) = 1/\sqrt{2}$

3) $G_M(x,t)$ satisfies the D.Eq.

$G''_M(x,t) = u_0(x) u_0(t) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$\Rightarrow G'_M(x,t) = \frac{x}{2} + A$

$\Rightarrow G_M(x,t) = \frac{x^2}{4} + Ax + B$

So for $[-1, t)$ and $(t, 1]$ we can write

$$G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & -1 \leq x < t \\ \frac{x^2}{4} + A'x + B' & t < x \leq 1 \end{cases}$$

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4) As $G_M(x,t)$ is cont. at $x=t$ so

$$G_M(t-0,t) = G_M(t+0,t)$$

$$\frac{t^2}{4} + At + B = \frac{t^2}{4} + A't + B'$$

$$\Rightarrow B = (A'-A)t + B'$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + (A'-A)t + B' \\ \frac{x^2}{4} + A'x + B' \end{cases}$$

⑤ By the end Point conditions

$$G_M(-1,t) = G_M(1,t) \Rightarrow \frac{1}{4} - A + (A'-A)t + B' = \frac{1}{4} + A' + B'$$

$$\Rightarrow -A + A't - At = +A' \Rightarrow A'(t-1) = A(t+1)$$

$$\Rightarrow A' = A\left(\frac{t+1}{t-1}\right)$$

and $G'_M(-1,t) = G'_M(1,t)$

$$\Rightarrow -\frac{1}{2} + A = \frac{1}{2} + A' = \frac{1}{2} + A\left(\frac{t+1}{t-1}\right)$$

$$\Rightarrow \left(1 - \frac{t+1}{t-1}\right) A = \frac{1}{2} + \frac{1}{2} = 1$$

$$\frac{-2}{t-1} A = 1 \Rightarrow A = -\frac{t-1}{2}$$

So $A' = -\frac{t-1}{2} \times \frac{t+1}{t-1} = -\frac{t+1}{2}$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} - \frac{t-1}{2}x - t + B' & -1 \le x < t \\ \frac{x^2}{4} - \frac{t+1}{2}x + B' & t < x \le 1 \end{cases}$$

⑥ Discont. cond. doesnot help to determine B'.

⑦ By the orthogonality condition

$$\int_{-1}^{t} \left(\frac{x^2}{4} - \frac{t-1}{2}x - t + B'\right)\left(\frac{1}{\sqrt{2}}\right)dx + \int_{t}^{1} \left(\frac{x^2}{4} - \frac{t+1}{2}x + B'\right)\frac{1}{\sqrt{2}}dx = 0$$

$$\frac{1}{\sqrt{2}}\left[\frac{t^3}{12} - \frac{t-1}{2}\cdot\frac{t^2}{2} - t^2 + B't + \frac{1}{12} + \frac{t-1}{4} + t + B' \right.$$
$$\left. + \frac{1}{12} - \frac{t+1}{4} + B' - \frac{t^3}{12} + \frac{t+1}{2}\cdot\frac{t^2}{2} - B't\right] = 0$$

$$B' = \frac{t^2}{4} + \frac{1}{6} + \frac{t}{2}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{4} - \frac{t-1}{2}x - \frac{t}{2} + \frac{t^2}{4} + \frac{1}{6}, & -1 \le x < t \\ \frac{x^2}{4} - \frac{t+1}{2}x + \frac{t^2}{4} + \frac{1}{6} + \frac{t}{2}, & t < x \le 1 \end{cases}$$

4410622

Example 4:-

Construct Green's Fun. associated with the system $u'' + \lambda u = 0$ with $u'(0)=0$ & $u(2)=0$

Sol.:-

1) First we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$

$u(2) = 0 \Rightarrow 2A + B = 0 \rightarrow *$

$u'(0) = 0 \Rightarrow A = 0$

And $* \Rightarrow 2(0) + B = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not eigenvalue

So Green's Function can be associated.

2) G.F. satisfies $G'' = 0 \Rightarrow G = Ax + B$

So
$$G(x,t) = \begin{cases} Ax + B & 0 \leq x < t \\ A'x + B' & t < x \leq 2 \end{cases}$$

③ As $G(x,t)$ is continuous on $[0,2]$, so is is also continuous at $x = t \in (0,2)$

$\Rightarrow G(t-0, t) = G(t+0, t)$

$At + B = A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \leq x < t \\ A'x + B' & t < x \leq 2 \end{cases}$$

④ $G(x,t)$ as a function of x satisfies the end point conditions

$G'(0,t) = 0$ $\quad A = 0$

$G(2,t) = 0 \Rightarrow A'(2) + B' = 0$

$\Rightarrow B' = -2A'$

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$$\Rightarrow G(x,t) = \begin{cases} A't - 2A' & 0 \le x < t \\ A'x - 2A' & t < x \le 2 \end{cases}$$

(5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$0 - A' = \frac{1}{1} \Rightarrow A' = -1$$

$$\Rightarrow G(x,t) = \begin{cases} 2-t & 0 \le x < t \\ 2-x & t < x \le 2 \end{cases}$$

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Solution of Boundary Value Problems Using Green's Functions.

Example:-

Solve the problem

$$u'' = f(x) \rightarrow (1)$$

with $u(0) = \alpha$, $u(l) = \beta \rightarrow (2)$

Sol.

Let $G(x, x')$ be the Green Function for the associated homogeneous B.V.P. Then it satisfies

$$\frac{d^2G}{dx^2} = \delta(x - x') \rightarrow (3)$$

$$G(0, x') = 0, \quad G(l, x') = 0 \rightarrow (4)$$

Then we can find that

$$G(x, x') = \begin{cases} -x(l - x')/l & 0 \le x < x' \\ -x'(l - x)/l & x' < x \le l \end{cases}$$

To find the relation $u(x)$ of (1) and $G(x, x')$ of (3) we use Lagrange's Identity with $v(x) = G(x, x')$. Then

$$\int_0^l [u(x) L(G) - G L(u)] \, dx = uG' - Gu' \Big|_0^l$$

$$= u(l) G'(l, x') - G(l, x') u'(l) - u(0) G'(0, x') + G(0, x') u'(0)$$

$$= u(l) G'(l, x') - (0) u'(l) - u(0) G'(0, x') + (0) u'(0)$$

$$= u(l) G'(l, x') - u(0) G'(0, x')$$

$$= \beta G'(l, x') - \alpha G'(0, x')$$

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$$= \beta\left(\frac{x'}{l}\right) - \alpha \frac{1}{l}(x' - l)$$

$$= \frac{\beta - \alpha}{l} x' + \alpha$$

$$\Rightarrow \int_0^l u(x)\, L(G)\, dx - \int_0^l G\, L(u)\, dx = \frac{\beta-\alpha}{l} x' + \alpha \quad \text{(5)}$$

Now $L(u) = \frac{d^2u}{dx^2} = f(x)$ ∵ here $L = \frac{d^2}{dx^2}$

and $L(G) = \frac{d^2G}{dx^2} = \delta(x - x')$ by (3)

So by eq. (5)

$$\int_0^l u(x)\,(\delta(x - x')\, dx - \int_0^l G\, f(x)\, dx = \frac{\beta-\alpha}{l} x' + \alpha$$

$$u(x') - \int_0^l G(x, x')\, f(x)\, dx = \frac{\beta-\alpha}{l} x' + \alpha$$

$$\Rightarrow u(x') = \int_0^l G(x, x')\, f(x)\, dx + \frac{\beta-\alpha}{l} x' + \alpha$$

$$\Rightarrow u(x) = \int_0^l G(x', x)\, f(x')\, dx' + \frac{\beta-\alpha}{l} x + \alpha$$

$$\Rightarrow u(x) = \int_0^l G(x, x')\, f(x')\, dx' + \frac{\beta-\alpha}{l} x + \alpha$$

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LAPLACE TRANSFORM AND ITS APPLICATIONS

Transformation:-

A Transformation is a Law or Rule which changes, sometime does not change, some variable or function into another variable or function.

Linear Transformation:-

A Transformation T is said to be linear if

$$T(\alpha_1 f_1(x) + \alpha_2 f_2(x)) = \alpha_1 T(f_1(x)) + \alpha_2 T(f_2(x))$$

Integral Transformation:-

There are several types of Transformations, but we are to concern only with Integral Transformation i.e. the transformation involving integrals.

Types of Integral Transforms.

The important & well known Integral transforms are

1) Laplace Transform
2) Fourier Transform
3) Mellin Transform
4) Hankel Transform
5) Kontrovich-Lebedev Transform
6) Mehler-Fock Transform.
7) Chebysheve Transform.

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DEFINITIN:-

Let $f(t)$ be a continuous or sectionally continuous function of t defined over $[0,\infty)$, then the Laplace transform of $f(t)$ is a function $F(s)$ of another variable s defined by

$$F(S) = \int_0^\infty e^{-st} f(t)\,dt$$

Small

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Remarks.

(1) The existance of Laplace transform of a function depends upon the existance of the defining integral.

(2) Every function may or may not has the Laplace Transform.

The Laplace Transform of a function $f(t)$, if it exists, is denoted by any one of the following notations $F(s)$, $\bar{F}(s)$, $\mathcal{L}\{f(t)\}$, $\mathcal{L}[f(t)]$ or $\mathcal{L}\{f(t);s\}$

Theorem:-

The Laplace Transformation operator $\mathcal{L}$ is a linear operator.

Proof:-

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \int_0^\infty e^{-st}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\}\,dt$$

$$= \int_0^\infty e^{-st}\{\alpha_1 f_1(t)\}\,dt + \int_0^\infty e^{-st}\{\alpha_2 f_2(t)\}\,dt$$

$$= \alpha_1 \int_0^\infty e^{-st} f_1(t)\,dt + \alpha_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= \alpha_1 \mathcal{L}\{f_1(t)\} + \alpha_2 \mathcal{L}\{f_2(t)\}$$

$\Rightarrow \mathcal{L}$ is linear operator.

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Definition:-

A function $F(t)$ is said to be of exponential order c if

$$|f(t)| \leq Me^{ct} \quad \text{for } t \geq t_0$$

Where M & t_0 are +ve constants and c is any constant

Functions of Class A.

A function which is piece-wise continuous and of some exponential order is said to be a function of class A.

THEOREM:-

Sufficient condition for the existance of Laplace Transform of a function is that it should be function of class A i.e. it should be piece-wise continuous and of exponential order.

PROOF:- Since f is of exponential order, so then by definition

$$|f(t)| \leq Me^{ct} \quad \text{for } t > t_0$$

Where M and t_0 are +ve constants while c is any constant.

Now $|f(t)| \leq Me^{ct}$

$$\Rightarrow |e^{-st} \cdot f(t)| \leq Me^{-st} \cdot e^{ct} \quad \text{for } t > t_0$$

$$= Me^{-(s-c)t} \quad \text{for } t > t_0$$

$$|L\{f(t)\}| = \left|\int_0^{\infty} e^{-st} f(t)\,dt\right| \leq M\left|\int_0^{\infty} e^{-(s-c)t}\,dt\right|$$

$$\leq \frac{M}{|s-c|} \quad \text{if } s >$$

Thus it is clear that $L\{f(t)\}$ exists.

Remark:-

1) The conditions described in the above theorem are sufficient but not necessary. e.g. the function $t^{-1/2}$ has Laplace Transform but is not piecewise continuous in any interval $[0, T]$, where $T > 0$.

2) From above if

$$\lim_{s\to\infty} F(s) \neq 0$$

then $F(s)$ can never be the L.T. of any function $f(t)$.

So we prove $\lim_{s\to\infty} F(s) = 0$.

Since $f(t)$ is piecewise cont. and is of exponential order, so then

$$|f(t)| \leq M e^{ct} \quad \forall \; t > t_0$$

Now

$$|F(s)| = \left|\int_0^\infty e^{-st} f(t)\, dt\right|$$

$$\leq \left|\int_0^\infty e^{-st} M e^{ct}\, dt\right|$$

$$= \left|\int_0^\infty e^{-(s-c)t} M\, dt\right|$$

$$= \frac{M}{s-c}$$

$$\left|\lim_{s\to\infty} F(s)\right| \leq \lim_{s\to\infty} \frac{M}{s-c} = 0$$

$$\lim_{s\to\infty} F(s) = 0$$

$L f(t) = F(s)$

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Laplace Transforms OF SOME FUNCTIONS

1) LAPLACE TRANSFORM OF A CONSTANT.

Let $f(t) = c$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}[c] = \int_0^\infty e^{-st} . c\, dt$

$= c \frac{e^{-st}}{-s}\Big|_0^\infty$

$= \frac{c}{-s}[0-1] = c/s$

$\Rightarrow \mathcal{L}\{c\} = c/s$

2) L.T. of t

$\mathcal{L}\{t\} = \int_0^\infty e^{-st} . t\, dt$

$= t . \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty (1) e^{-st}/_{-s}\, dt$

$= (0-0) + \frac{1}{s}\int_0^\infty 1 . e^{-st}\, dt$

$= 0 + \frac{1}{s} . (\frac{1}{s}) = \frac{1}{s^2}$

$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$

3) L.T. of t^2

$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt$

$= t^2 . \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 2t . \frac{e^{-st}}{-s}\, dt$

$= 0 - 0 + \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s} . \frac{1}{s^2}$

$= \frac{2.1}{s^3} = \frac{2!}{s^3}$

4) L.T. of t^3

$\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st}\, dt$

$= t^3 \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 3t^2 \frac{e^{-st}}{-s}\, dt$

$= 0 + \frac{3}{s}\mathcal{L}\{t^2\}$

$= \frac{3}{s} . \frac{2!}{s^3} = \frac{3!}{s^4}$

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5) L.T. of t^n

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$$

$$= t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{n}{s}\cdot \mathcal{L}\{t^{n-1}\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot \mathcal{L}\{t^{n-2}\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdot \mathcal{L}\{t^{n-3}\}$$

$$= \frac{n(n-1)(n-2)}{s^3}\cdots\cdots\frac{2}{s}\cdot\frac{1}{s}\cdot\mathcal{L}\{t^0\}$$

$$= \frac{n!}{s^n}\cdot\frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

L.T. of e^{kt}

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}\cdot e^{kt}\,dt$$

$$= \int_0^\infty e^{-(s-k)t}\,dt$$

Let $s-k = s'$ then

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-s't}\,dt = \int_0^\infty 1\cdot e^{-s't}\,dt = \frac{1}{s'}$$

$$\Rightarrow \mathcal{L}\{e^{kt}\} = \frac{1}{s-k} \quad \text{for } s>k$$

L.T of Sinkt

$$\mathcal{L}\{\sin kt\} = \mathcal{L}\left\{\frac{e^{ikt}-e^{-ikt}}{2i}\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{s-ik} - \frac{1}{s+ik}\right] = \frac{1}{2i}\left[\frac{s+ikt-s+ikt}{s^2-(ik)^2}\right]$$

$$= \frac{k}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

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L.T. of Coskt.

$$\mathcal{L}\{\cos kt\} = \mathcal{L}\left\{\frac{e^{ikt}+e^{-ikt}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-ik} + \frac{1}{s+ik}\right]$$

$$= \frac{1}{2}\left[\frac{s+ik+s-ik}{s^2+k^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2+k^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

L.T. of Sinhkt.

$$\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt}-e^{-kt}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$

$$= \frac{1}{2}\left[\frac{s+k-s+k}{s^2-k^2}\right] = \frac{k}{s^2-k^2}$$

L.T. of Coshkt.

$$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right] = \frac{1}{2}\left[\frac{s+k+s-k}{s^2-k^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

The First Shifting Theorem.

Statement.

$$\mathcal{L}\{e^{kt}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-k} = F(s)\big|_{s\to s-k} = F(s-k)$$

Proof.

$$\mathcal{L}\{e^{kt}f(t)\} = \int_0^\infty e^{-st}\cdot e^{kt}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-k)t}f(t)\,dt$$

$$= \int_0^\infty e^{-s't}f(t)\,dt \quad \text{where } s' = s-k$$

$$= F(s')$$

$$= F(s-k)$$

$$= F(s)\big|_{s\to s-k}$$

It is also called 1st Translation Theorem

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آئیکسی سائنیز

1) Find L.T. of $\sin^2 \omega t$

$$\mathcal{L}\{\sin^2 \omega t\} = \mathcal{L}\left\{\frac{1-\cos 2\omega t}{2}\right\}$$
$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2\omega t\}$$
$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4\omega^2}$$

$$\Rightarrow \mathcal{L}\{\sin^2 \omega t\} = \frac{1}{2s} - \frac{s}{2(s^2+4\omega^2)}$$
$$= \frac{s^2+4\omega^2-s^2}{2s(s^2+4\omega^2)}$$
$$= \frac{2\omega^2}{s(s^2+4\omega^2)}$$

2) Find L.T. of $\cos^2 \omega t$

$$\mathcal{L}\{\cos^2 \omega t\} = \mathcal{L}\left\{\frac{1+\cos 2\omega t}{2}\right\}$$
$$= \frac{1}{2s} + \frac{s}{2(s^2+4\omega^2)}$$
$$= \frac{s^2+4\omega^2+s^2}{2s(s^2+4\omega^2)} = \frac{2(s^2+2\omega^2)}{2s(s^2+4\omega^2)}$$

$$\mathcal{L}\{\cos^2 \omega t\} = \frac{s^2+2\omega^2}{s(s^2+4\omega^2)}$$

3) $\mathcal{L}\{\sin(\omega t-\phi)\}$

$$\mathcal{L}\{\sin(\omega t-\varphi)\} = \mathcal{L}\left\{\frac{e^{i(\omega t-\varphi)} - e^{-i(\omega t-\varphi)}}{2i}\right\}$$

$$= \frac{1}{2i}\left[\mathcal{L}\{e^{i\omega t}\cdot e^{-i\varphi}\} - \mathcal{L}\{e^{-i\omega t}\cdot e^{i\varphi}\}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-i\varphi}}{s}\bigg|_{s\to s-i\omega} - \frac{e^{i\varphi}}{s}\bigg|_{s\to s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-i\varphi}}{s-i\omega} - \frac{e^{i\varphi}}{s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{\cos\varphi - i\sin\varphi}{s-i\omega} - \frac{\cos\varphi + i\sin\varphi}{s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{s\cos\varphi - is\sin\varphi + i\omega\cos\varphi + \omega\sin\varphi - s\cos\varphi - is\sin\varphi + i\omega\cos\varphi - \omega\sin\varphi}{s^2+\omega^2}\right]$$

$$= \frac{\omega\cos\phi - s\sin\phi}{s^2+\omega^2}$$

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4) L.T. of $\cos(\omega t - \phi)$

$$\mathcal{L}\{\cos(\omega t-\varphi)\} = \mathcal{L}\left\{\frac{e^{i(\omega t-\varphi)}+e^{-i(\omega t-\varphi)}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{\cos\varphi - i\sin\varphi}{s-i\omega} + \frac{\cos\varphi + i\sin\varphi}{s+i\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s\cos\varphi - is\sin\varphi + i\omega\cos\varphi + \omega\sin\varphi - i\omega\cos\varphi + \omega\sin\varphi + s\cos\varphi + is\sin\varphi}{(s-i\omega)(s+i\omega)}\right]$$

$$= \frac{s\cos\varphi + \omega\sin\varphi}{s^2+\omega^2}$$

5) L.T. of $e^{2(t+1)}$

$$\mathcal{L}\{e^{2(t+1)}\} = \mathcal{L}\{e^{2t}\cdot e^{2}\}$$

$$= \mathcal{L}\{e^{2}\}_{s\to s-2}$$

$$= \frac{e^2}{s}\Big|_{s\to s-2} = \frac{e^2}{s-2}$$

6) $\mathcal{L}\{\sin\omega t\, e^{-2t}\}$

$$\mathcal{L}\{\sin\omega t\, e^{-2t}\} = \mathcal{L}\{\sin\omega t\}_{s\to s+2}$$

$$= \frac{\omega}{s^2+\omega^2}\Big|_{s\to s+2}$$

$$= \frac{\omega}{(s+2)^2+\omega^2}$$

$$\Rightarrow \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+4s+4+\omega^2}$$

7) L.T. of $\cos 3\omega t\, e^{4t}$

$$\mathcal{L}\{\cos 3\omega t\, e^{4t}\} = \frac{s}{(s-4)^2+9\omega^2}$$

8) L.T. of $e^{3t}\, t^4$

$$\mathcal{L}\{e^{3t}\, t^4\} = \frac{4!}{s^5}\Big|_{s\to s-3}$$

$$= \frac{24}{(s-3)^5}$$

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8) Calculate $\mathcal{L}\{t^\alpha\}$ where $\alpha \in \mathbb{R}$

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st} t^\alpha \, dt$$

put $st = u$ $\quad sdt = du \Rightarrow dt = \frac{1}{s}du$

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \cdot \frac{1}{s} du$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha du$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^{\alpha+1-1} du$$

$$= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Here we have used the def. of Gamma Function defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$$

9) L.T. of $t^{1/2}$

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma(3/2)}{s^{3/2}}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} \quad \because \Gamma n = (n-1)\Gamma n-1$$

$$= \frac{\sqrt{\pi}}{2 s^{3/2}} \quad \because \Gamma(1/2) = \sqrt{\pi}$$

10) L.T. of $t^{-1/2}$

$$\text{L.T. of } t^{-1/2} = \mathcal{L}\{t^{-1/2}\}$$

$$= \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{\Gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

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Question :- Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Sol.

Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

Put $x = r\cos\theta$ $\quad y = r\sin\theta$

$dx\,dy = J\,dr\,d\theta = r\,dr\,d\theta$

and $0 \le r < \infty$ $\quad 0 \le \theta \le 2\pi$

$$\Rightarrow I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\,dr\,d\theta$$

$$= \left[-\frac{1}{2}\int_0^{\infty} e^{-r^2}(-2r)dr\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= -\frac{1}{2}\left(e^{-r^2}\right)\Big|_0^{\infty} \cdot \theta\Big|_0^{2\pi}$$

$$= -\frac{1}{2}(0-1)(2\pi - 0) = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

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Question:-

Use the result

$$\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\}, \quad s>0, \quad \alpha > -1$$

Calculate

$$\mathcal{L}\{t^{k/2}\}$$

Where k is an odd +ve integer.

Sol. Since k is an odd +ve integer so it can be written as

$$k = 2m+1 \qquad m \in W$$

$$\Rightarrow t^{k/2} = t^{\frac{2m+1}{2}} = t^{m+1/2}$$

$$\mathcal{L}\{t^{m+\frac{1}{2}}\} = \frac{m+\frac{1}{2}}{s}\,\mathcal{L}\{t^{m-\frac{1}{2}}\}$$

$$= \frac{m+\frac{1}{2}}{s}\cdot\frac{m-\frac{1}{2}}{s}\cdot\mathcal{L}\{t^{m-3/2}\}$$

$$= \frac{m+\frac{1}{2}}{s}\cdot\frac{m-\frac{1}{2}}{s}\cdot\frac{m-\frac{3}{2}}{s}\cdot\frac{m-5/2}{s}\cdots\cdots\frac{3/2}{s}\cdot\frac{1/2}{s}\cdot\mathcal{L}\{t^{-1/2}\}$$

$$= \frac{2m+1}{2s}\cdot\frac{2m-1}{2s}\cdot\frac{2m-3}{2s}\cdots\cdots\frac{3}{2s}\cdot\frac{1}{2s}\cdot\sqrt{\frac{\pi}{s}}$$

$$= \frac{k(k-2)(k-4)\cdots 3\cdot 1}{(2s)^{(k+1)/2}}\cdot\frac{\sqrt{\pi}}{\sqrt{s}}$$

$a_n = a_1 + (n-1)d$
$k = 1 + (n-1)d$
$n = \frac{k+1}{2}$

$$\Rightarrow \mathcal{L}\{t^{k/2}\} = \frac{k(k-2)(k-4)\cdots 3\cdot 1}{2^{k/2+1/2}}\cdot\sqrt{\frac{\pi}{s^{k+2}}}$$

Question:-

Derive L.T. of $t^{1/2}$ and $t^{-1/2}$ from definition.

Sol.

$$\mathcal{L}\{t^{1/2}\} = \int_0^\infty e^{-st}\, t^{1/2}\,dt$$

Put $st = u$ $\quad dt = \frac{1}{s}du$ and $t = u/s$

$$\Rightarrow \mathcal{L}\{t^{1/2}\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^{1/2}\cdot\frac{1}{s}du$$

$$= \frac{1}{s^{3/2}}\int_0^\infty e^{-u}\,u^{1/2}\,du$$

Put $u^{1/2} = x \Rightarrow u = x^2$

$\Rightarrow du = 2x\,dx$

$$\Rightarrow \mathcal{L}\{t^{1/2}\} = \frac{1}{s^{3/2}}\int_0^\infty e^{-x^2}\,x\cdot 2x\,dx$$

$$= \frac{1}{s^{3/2}}(-1)\int_0^\infty x\left(e^{-x^2}(-2x)\right)dx$$

$$= \frac{1}{s^{3/2}}(-1)\left[x\cdot e^{-x^2}\Big|_0^\infty - \int_0^\infty 1\cdot e^{-x^2}dx\right]$$

4410622

$= \frac{1}{s^{3/2}} \int_0^\infty e^{-x^2} dx \quad = \frac{1}{s^{3/2}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx$

$= \frac{1}{2 \cdot s^{3/2}} \sqrt{\pi} = \frac{\sqrt{\pi}}{2 s^{3/2}}$

Now $\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st} t^{-1/2} dt$

Put $st = u$

$\Rightarrow \mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^{-1/2} \cdot \frac{1}{s} du$

$= \frac{1}{s^{1/2}} \int_0^\infty e^{-u} u^{-1/2} du$

Put $u = x^2 \qquad du = 2x\,dx$

$\Rightarrow \mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{1}{s^{1/2}} \int_0^\infty e^{-x^2} \cdot x^{-1} \cdot 2x\,dx$

$= \frac{1}{s^{1/2}} \int_{-\infty}^{\infty} e^{-x^2} dx$

$= \frac{1}{s^{1/2}} \cdot \sqrt{\pi}$

$\Rightarrow \mathcal{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}$

Q. Prove that.

1) $\Gamma(x+1) = x\Gamma(x)$ (2) $\Gamma(1) = 1$

Sol.

$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x+1-1} dt = t^x e^{-t}\Big|_0^\infty - \int_0^\infty x t^{x-1} \frac{e^{-t}}{-1} dt$

$= 0 + x \int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x)$

$\Rightarrow \Gamma(x+1) = x\Gamma(x)$

Further $\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = x(x-1)(x-2)\Gamma(x-2)$

$= x(x-1)(x-2)\cdots 3\cdot 2\cdot 1\,\Gamma(1)$

$= x(x-1)(x-2)\cdots 3\cdot 2\cdot 1 = x!$ if $x \in Z^+$

$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = -(0-1) =$

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LAPLACE TRANSFORMS OF Derivatives And Integrals

Theorem:-

If $f(t), f'(t), f''(t), \ldots, f^{(n-1)}(t)$ are continuous and $f^{(n)}(t)$ is piecewise continuous on the interval $[0,\infty)$ and all are of exponential order, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Find the case when $n=2$

Proof:-

We prove the theorem by the Principle of Mathematical Induction

For $n=1$.

$$L.H.S. = \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} . f(t)\,dt$$

$$= -f(0) + s \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\, F(s)$$

$$= s\,F(s) - f(0)$$

$$R.H.S. = s^1 F(s) - s^{1-1} f(0)$$

$$= s\,F(s) - f(0)$$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ Theorem is true for $n=1$.

So C-1 is satisfied.

Now suppose that theorem is true for $n=k$ i.e.

$$\mathcal{L}\{f^{(k)}(t)\} = s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - s^{k-3} f''(0) - \ldots - s f^{k-2}(0) - f^{k-1}(0) \longrightarrow (1)$$

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Now we prove the theorem for $n=k+1$. For this

$$\mathcal{L}\{f^{(k+1)}(t)\} = \int_0^\infty e^{-st} f^{(k+1)}(t)\,dt$$

$$= e^{-st} f^{(k)}(t)\Big|_0^\infty - \int_0^\infty (-s)\, e^{-st} f^{(k)}(t)\,dt$$

$$= -f^{(k)}(0) + s\,\mathcal{L}\{f^{(k)}(t)\}$$

$$= -f^{(k)}(0) + s\{s^k F(s) - s^{k-1}f(0) - s^{k-2}f'(0) - s^{k-3}f''(0) - \cdots - s f^{k-2}(0) - f^{k-1}(0)\}$$ by ①

$$= s^{k+1}F(s) - s^{k+1-1}f(0) - s^{k+1-2}f'(0) - s^{k+1-3}f''(0) - \cdots - s^2 f^{k+1-3}(0) - s f^{k+1-2}(0) - f^{k+1-1}(0)$$

⇒ Theorem is true for $n=k+1$

⇒ C-2 is satisfied.

⇒ Theorem is true upto the ~~nth~~ ~~derivative~~ ~~of~~ Laplace of nth derivative of $f(t)$.

Now if $n=2$, then

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

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Theorem:-

Prove that

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Proof:-

Let $g(t) = \int_0^t f(\tau)\,d\tau$

Then by the Fundamental Theorem of Integral Calculus

$$g'(t) = f(t)$$

Then $\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\}$

$$s\mathcal{L}\{g(t)\} - g(0) = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s\mathcal{L}\{g(t)\} - \int_0^0 f(\tau)\,d\tau = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - 0 = \mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Q:-

Calculate $\mathcal{L}\left\{\int_0^t e^{kt}\cos\omega t\,dt\right\}$

Sol.

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$$\mathcal{L}\left\{\int_0^t e^{kt}\cos\omega t\right\}$$

$$= \frac{1}{s}\mathcal{L}\{e^{kt}\cos\omega t\}$$

$$= \frac{1}{s}\left[\left(\frac{s}{s^2+\omega^2}\right)\Big|_{s\to s-k}\right]$$ By 1st shifting Theorem

$$= \frac{1}{s}\cdot\frac{s-k}{(s-k)^2+\omega^2}$$

$$= \frac{s-k}{s\left((s-k)^2+\omega^2\right)}$$

INVERSE LAPLACE TRANSFORM.

If F(s) is the Laplace Transform of f(t) then f(t) is called the inverse Laplace Transform of F(s).

The inverse L.T. of F(s) is denoted by

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

The following Table shows the relation b/w F(s) and $\mathcal{L}^{-1}\{F(s)\}$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
c/s	c , $s>0$
$n!/s^{n+1}$	t^n , $s>0$
Here $n \in Z^+$	
$1/s-k$	e^{kt} $s>k$
k/s^2+k^2	$\sin kt$ $s>0$
s/s^2+k^2	$\cos kt$ $s>0$
k/s^2-k^2	$\sinh kt$ $s>\lvert k\rvert$
s/s^2-k^2	$\cosh kt$ $s>\lvert k\rvert$
$F(s-k)$	$e^{kt}\mathcal{L}^{-1}\{F(s)\}$ $s>k$
$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)$ ------ $f^{n-1}(0)$	$f^{(n)}(t)$
$\frac{1}{s}\cdot F(s)$	$\int_0^t f(\tau)\,d\tau$
$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$\mathcal{L}\{t^\alpha\}$, $\alpha \in R$
$\frac{1}{s}\sqrt{\pi/s}$	$2t^{1/2}$
$\sqrt{\pi/s}$	$t^{-1/2}$

First Shifting Theorem For Inverse L.T.

$$\mathcal{L}^{-1}\{F(s-k)\} = e^{kt}\,\mathcal{L}^{-1}\{F(s)\}$$

Example: Find I.L.T. of

(i) $\frac{1}{s^2+2s}$ (ii) $\frac{s}{s^2+2s}$

Sol.

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-1+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{[s-(-1)]^2-1}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-1^2}\right\} = e^{-t}\sinh t$

$= \cancel{e^{t}}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$ By 1st shifting Theorem

$= \cancel{e^{t}\sin t}$

(ii)

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s(s+2)}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

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Use of Partial Fraction in Calculating Inverse Laplace Transform.

Exp.

Calculate $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Sol.

Consider. $\frac{s+4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

Xing both sides by $(s+1)(s+2)$

$s+4 = A(s+2) + B(s+1)$

Put $s=-1$

$3 = A(-1+2)+0 \Rightarrow \boxed{A=3}$

put $s=-2$

$-2+4 = A(0) + B(-2+1)$

$\Rightarrow B=-2$

$\Rightarrow \frac{s+4}{s^2+3s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$

$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{s+2}\right\}$

$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= 3e^{-t} - 2e^{-2t}$

APPLICATIONS OF L.T. TO I.V.P.

Before applying L.T. to I.V.Ps. the following results must be kept in view

(i) $\mathcal{L}\{u(t)\} = U(s)$

(ii) $\mathcal{L}\{u'(t)\} = sU(s) - u(0)$

(iii) $\mathcal{L}\{u''(t)\} = s^2 U(s) - su(0) - u'(0)$

(iv) $\mathcal{L}\{u'''(t)\} = s^3 U(s) - s^2 u(0) - su'(0) - u''(0)$

Exp. 1:- Solve the I.V.P.

$$u' - 2u = 0 \quad , \quad u(0) = 1$$

Sol.

Given $u' - 2u = 0$

$$\mathcal{L}\{u' - 2u\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0$$

$$\Rightarrow sU(s) - u(0) - 2U(s) = 0$$

$$\Rightarrow sU(s) - 1 - 2U(s) = 0$$

$$\Rightarrow (s-2)U(s) = 1$$

$$\Rightarrow U(s) = \frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$u(t) = e^{2t}$$

Exp. 2:- Solve the I.V.P.

$$u' + 2u = 0 \quad , \quad u(0) = 1$$

Sol.

$$\mathcal{L}\{u'\} + 2\mathcal{L}\{u\} = \mathcal{L}\{0\}$$

$$sU(s) - u(0) + 2U(s) = 0$$

$$(s+2)U(s) = 1$$

$$\Rightarrow U(s) = \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$u(t) = e^{-2t}$$

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Exp. 3:- Solve

$$u'' + 4u' + 3u = 0 \qquad u(0) = 1,\ u'(0) = 0$$

Sol.

Taking L.T. on both sides

$$s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = 0$$

$$s^2U(s) - s(1) - 0 + 4sU(s) - 4(1) + 3U(s) = 0$$

$$(s^2 + 4s + 3)U(s) = s + 4$$

$$\Rightarrow U(s) = \frac{s+4}{(s+3)(s+1)} = \frac{-1/2}{s+3} + \frac{3/2}{s+1}$$

$$u(t) = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}$$

Exp. 4:- Solve

$$u'' + 9u = 0, \quad u(0) = 0,\ u'(0) = 1$$

Sol.

$$\mathcal{L}\{u''\} + 9\mathcal{L}\{u\} = \mathcal{L}\{0\}$$

$$s^2U(s) - su(0) - u'(0) + 9U(s) = 0$$

$$(s^2+9)U(s) - s(0) - 1 = 0$$

$$U(s) = \frac{1}{s^2+3^2}$$

$$U(s) = \frac{1}{3}\cdot\frac{3}{s^2+3^2}$$

$$\Rightarrow u(t) = \frac{1}{3}\sin 3t$$

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UNIT STEP FUNCTION.

Unit step Function or Heaviside step function is denoted and defined by

$$H(t-t_0) = \begin{cases} 0 & \text{if } t < t_0 \\ 1 & \text{if } t \geq t_0 \end{cases}$$

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کان وُو لوشن

Convolution.

If $f(t)$ and $g(t)$ are piecewise continuous functions over the interval $[0, \infty[$, then their convolution denoted by $f*g$ and is defined by

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Theorem:-

If f, g and h are piecewise continuous functions over $[0, \infty[$, then

(i) $f*g = g*f$

(ii) $f*(g+h) = f*g + f*h$

(iii) $f*(g*h) = (f*g)*h$

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Proof:-

(i)-

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

put $t-\tau = T$ $\quad -d\tau = dT$

$\tau = 0 \Rightarrow T = t$

$\tau = t \Rightarrow T = 0$

So $f*g = \int_t^0 f(t-T)\, g(T)\, (-dT)$

$$= \int_0^t g(T)\, f(t-T)\, dT$$

$$= g*f$$

$$\Rightarrow f*g = g*f$$

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ii) $f*(g+h) = \int_0^t f(\tau)\,(g(t-\tau)+h(t-\tau))\,d\tau$

$= \int_0^t f(\tau)g(t-\tau)\,d\tau + \int_0^t f(\tau)h(t-\tau)\,d\tau$

$= f*g + f*h$

iii) To prove $f*(g*h) = (f*g)*h$

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Corollary:- Prove that

$$f*g = \int_0^\infty H(t-\tau)\, f(\tau)\, g(t-\tau)\, d\tau$$

Where H is Unit Step Function.

Proof:-

$$R.H.S. = \int_0^\infty H(t-\tau)\, f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t H(t-\tau) f(\tau)\, g(t-\tau)\, d\tau + \int_t^\infty H(t-\tau)\, f(\tau) g(t-\tau)\, d\tau$$

Here $H(t-\tau) = \begin{cases} 0 & \text{if } t < \tau \\ 1 & \text{if } t \geq \tau \end{cases}$

So $R.H.S. = \int_0^t (1) f(\tau) g(t-\tau) d\tau + \int_t^\infty (0)\, f(\tau)\, g(t-\tau)\, d\tau$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau + 0$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= f*g$$

$$\Rightarrow f*g = \int_0^\infty H(t-\tau)\, f(\tau)\, g(t-\tau)\, d\tau$$

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Convolution Theorem.

If $f(t)$ and $g(t)$ are piecewise cont. functions over $[0,\infty[$ and are of exponential order, then

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = f*g$$

Proof:-

To prove: $\mathcal{L}^{-1}\{F(s)\, G(s)\} = f*g$

i.e. to prove $\mathcal{L}\{f*g\} = F(s)\, G(s)$

Now

$$\mathcal{L}\{f*g\} = \int_0^\infty e^{-st} f*g\, dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right) dt$$

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$$= \int_0^\infty e^{-st} \int_0^t f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty e^{-st} \int_0^\infty H(t-\tau) f(\tau) g(t-\tau)\, d\tau\, dt$$

Where $H(t-\tau)$ is unit step function defined by

$$H(t-\tau) = \begin{cases} 0 & , \text{if } t < \tau \\ 1 & , \text{if } t \geq \tau \end{cases}$$

$$\Rightarrow \mathcal{L}\{f*g\} = \int_0^\infty \left[\int_0^\infty e^{-st} H(t-\tau)\, g(t-\tau)\, dt \right] f(\tau)\, d\tau$$

Put $t-\tau = T$

$t=0 \Rightarrow T = -\tau$

$t = \infty \Rightarrow T = \infty$

$dt = dT$

$$\mathcal{L}\{f*g\} = \int_0^\infty \left[\int_{-\tau}^\infty e^{-s(T+\tau)} H(T)\, g(T)\, dT \right] f(\tau)\, d\tau$$

$$= \int_0^\infty \left[\int_{-\tau}^0 e^{-sT} H(T)\, g(T)\, dT + \int_0^\infty e^{-sT} H(T)\, g(T)\, dT \right] e^{-s\tau} f(\tau)\, d\tau$$

$$= \int_0^\infty \left[0 + \int_0^\infty e^{-sT} (1)\, g(T)\, dT \right] e^{-s\tau} f(\tau)\, d\tau$$

$$= \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau \right) \left(\int_0^\infty e^{-sT} g(T)\, dT \right)$$

$$= F(s)\, G(s)$$

$$\Rightarrow \mathcal{L}\{f*g\} = F(s)\, G(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\, G(s)\} = f*g$$

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Examples:

Exp. 1:- Calculate

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+k^2)}\right\}$ by convolution Theorem.

Sol.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+k^2}\right\}$$

$$= \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\cdot\frac{1}{s}\right\}$$

$$= \frac{1}{k}\left[\sin kt * 1\right] \quad \text{By convolution Theorem.}$$

$$= \frac{1}{k}\cdot\int_0^t \sin k\tau\cdot(t-\tau)^0\, d\tau \qquad \because 1 = t^0$$

$$= \frac{1}{k}\int_0^t \sin k\tau\, d\tau$$

$$= \frac{1}{k}\cdot\left[-\frac{\cos k\tau}{k}\right]\Big|_0^t$$

$$= \frac{1}{k^2}\left[-\cos kt + \cos(0)\right] = \frac{1}{k^2}\left[1-\cos kt\right]$$

Exp. 2:- Calculate L.T. of

$\int_0^t (t-\beta)\sin 3\beta\, d\beta$

Sol.

$$\mathcal{L}\left\{\int_0^t \sin 3\beta\,(t-\beta)\,d\beta\right\} = \mathcal{L}\left\{\sin 3t * t\right\}$$

$$= \mathcal{L}\{\sin 3t\}\,\mathcal{L}\{t\}$$

$$= \frac{3}{s^2+9}\cdot\frac{1}{s^2}$$

$$= \frac{3}{s^2(s^2+9)}$$

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Q1. Find the L.T. of

a) $\int_0^t e^{-(t-\beta)} \sin\beta \, d\beta$

b) $\int_0^t (t-\beta)^3 \sin\beta \, d\beta$

Sol.

a) $\mathcal{L}\{\int_0^t \sin\beta \, e^{-(t-\beta)} d\beta\}$

$= \mathcal{L}\{\sin t * e^{-t}\}$

$= \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{e^{-t}\}$

$= \frac{1}{s^2+1^2} \cdot \frac{1}{s+1} = \frac{1}{(s+1)(s^2+1)}$

b) $\mathcal{L}\{\int_0^t (t-\beta)^3 \sin\beta \, d\beta\}$

$= \mathcal{L}\{\sin t * t^3\}$

$= \frac{1}{s^2+1} \cdot \frac{3!}{s^4} = \frac{6}{s^4(s^2+1)}$

Q2. Find I.L.T. by convolution Theorem

a) $\frac{4}{s^2(s-2)}$ b) $\frac{1}{(s^2+1)^2}$

Sol.

a) $\mathcal{L}^{-1}\left\{\frac{4}{s^2(s-2)}\right\}$

$= 4\mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{s-2}\right\}$

$= 4 \cdot [t * e^{2t}]$

$= 4\int_0^t \mathcal{T} e^{2(t-\mathcal{T})} d\mathcal{T}$

$= 4e^{2t}\left[\mathcal{T}\frac{e^{-2\mathcal{T}}}{-2}\Big|_0^t - \int_0^t \frac{e^{-2\mathcal{T}}}{-2} d\mathcal{T}\right]$

$= 4e^{2t}\left[-\frac{1}{2}te^{-2t} - 0 + \frac{1}{2} \cdot \frac{e^{-2\mathcal{T}}}{-2}\Big|_0^t\right]$

$= 4e^{2t}\left[-\frac{1}{2}te^{-2t} - \frac{1}{4}[e^{-2t} - e^0]\right]$

4410622

$= 4e^{2t}\left[-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right]$

$= -2t - 1 + e^{2t}$

$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{4}{s^2(s-2)}\right\} = e^{2t} - 2t - 1$

b) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \frac{1}{s^2+1}\right\}$

$= \sin t * \sin t$

$= \int_0^t \sin \tau \sin(t-\tau)\, d\tau$

$= -\frac{1}{2}\int_0^t (-2\sin\tau \sin(t-\tau))\, d\tau$

$= -\frac{1}{2}\int_0^t [\cos(\tau + t - \tau) - \cos(\tau - t + \tau)]\, d\tau$

$= -\frac{1}{2}\left[\int_0^t \cos t\, d\tau - \int_0^t \cos(2\tau - t)\, d\tau\right]$

$= -\frac{1}{2}\left[t \sin t - \frac{\sin(2\tau - t)}{2}\Big|_0^t\right]$

$= -\frac{1}{2}\left[t\sin t - \frac{1}{2}(\sin t + \sin t)\right]$

$= -\frac{1}{2}[t\sin t - \sin t]$

$= \frac{\sin t}{2} - \frac{t \sin t}{2}$

Q 3. Show that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau$

Sol.

$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{F(s) \cdot \frac{1}{s}\right\}$

$= f(t) * 1 = f(t) * t^0 = \int_0^t f(\tau)(t-\tau)^0\, d\tau$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau \qquad \because (t-\tau)^0 = 1$

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Q4:- Show that
$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\} d\xi$$

Sol.

$$\text{L.H.S.} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\cdot\frac{1}{s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} * 1$$
$$= \mathcal{L}^{-1}\left\{F(s) * 1/s\right\} * 1$$
$$= (f(t) * 1) * 1$$
$$= \left(\int_0^t f(\lambda)\,d\lambda\right) * 1$$
$$= \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\}(t-\xi)^0\,d\xi$$
$$= \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\} d\xi = \text{R.H.S.}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\} d\xi$$

Q5. a) Show that
$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = t\int_0^t f(\lambda)\,d\lambda - \int_0^t t f(t)\,dt$$

b) Calculate
$$\mathcal{L}\{\ln t\}$$

Sol.

a) $$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\} d\xi$$
$$= \int_0^t \left\{\int_0^{\xi} f(\lambda)\,d\lambda\right\}\cdot 1\,d\xi$$
$$= \left(\int_0^{\xi} f(\lambda)\,d\lambda\right)\xi\Big|_0^t - \int_0^t f(\xi)\cdot\xi\,d\xi \quad \text{(Integrating by parts)}$$
$$= t\int_0^t f(\lambda)\,d\lambda - \int_0^t \xi f(\xi)\,d\xi$$
$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = t\int_0^t f(\lambda)\,d\lambda - \int_0^t \xi f(\xi)\,d\xi$$

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b) To Calculate $\mathcal{L}\{\ln t\}$

$\mathcal{L}\{\ln t\} = \int_0^\infty e^{-st} \ln t \, dt$

Put $st = u$ $\Rightarrow \mathcal{L}\{\ln t\} = \int_0^\infty e^{-u} (-\ln s + \ln u) \frac{du}{s}$

$= -\frac{\ln s}{s} \int_0^\infty e^{-u} du + \frac{1}{s} \int_0^\infty e^{-u} \ln u \, du$

$= -\frac{\ln s}{s} \cdot 1 + \frac{1}{s} \int_0^\infty e^{-u} \ln u \, du$

Now $\Gamma(x+1) = \int_0^\infty e^{-u} u^x du$

$\Gamma'(x+1) = \int_0^\infty e^{-u} u^x \ln u \, du \Rightarrow \Gamma'(1) = \int_0^\infty e^{-u} \ln u \, du$

$\Rightarrow \mathcal{L}\{\ln t\} = -\frac{\ln s}{s} + \frac{1}{s} \Gamma'(1)$

$\Rightarrow \mathcal{L}\{\ln t\} = \frac{1}{s}(\Gamma'(1) - \ln s)$

Q.6. Same as Q.1.

Q.7. Solve the Integral eq.

$y(t) = at + \int_0^t y(\tau) \sin(t-\tau) d\tau$

Sol.

Let $u(t) = \int_0^t y(\tau) \sin(t-\tau) d\tau$

$= y(t) * \sin t$

$\mathcal{L}\{u(t)\} = \mathcal{L}\{y(t) * \sin t\}$

$= \mathcal{L}\{y(t)\} \mathcal{L}\{\sin t\}$

$U(s) = Y(s) \cdot \frac{1}{s^2+1}$

Now $y(t) = at + u(t)$

$\mathcal{L}\{y(t)\} = \mathcal{L}\{at\} + \mathcal{L}\{u(t)\}$

$\Rightarrow Y(s) = a/s^2 + U(s) \Rightarrow Y(s) = a/s^2 + Y(s) \cdot \frac{1}{s^2+1}$

$\Rightarrow Y(s)\left(1 - \frac{1}{s^2+1}\right) = a/s^2$

$\Rightarrow Y(s)\left[\frac{s^2+1-1}{s^2+1}\right] = a/s^2 \Rightarrow Y(s) = a/s^2 \cdot \frac{s^2+1}{s^2}$

$\Rightarrow Y(s) = a\left(\frac{s^2+1}{s^4}\right) = a\left(\frac{1}{s^2} + \frac{1}{s^4}\right) = \frac{a}{s^2} + \frac{a}{3!} \cdot \frac{3!}{s^4}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{a}{s^2} + \frac{a}{3!} \cdot \frac{3!}{s^4}\right\}$

$\Rightarrow y(t) = at + \frac{a}{3!} t^3$

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Q8:- Solve

$$y(t) = f(t) + \int_0^t g(t-\tau)\, y(\tau)\, d\tau$$

Sol.

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t y(\tau)\, g(t-\tau)\, d\tau\right\}$$

$$Y(S) = F(S) + Y(S)\,G(S) \rightarrow$$ By Convolution Theorem

$$Y(S)[1 - G(S)] = F(S)$$

$$\Rightarrow Y(S) = \frac{F(S)}{1 - G(S)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{F(S)}{1 - G(S)}\right\}$$

Q9:- Solve the D. Eqs. by L. T. Method.

a) $y''(t) + k^2 y(t) = f(t)$

b) $y''(t) - 2k y'(t) + k^2 y(t) = f(t)$

c) $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$

{ In each case discuss the Physical Significance }

Sol.

a) $y''(t) + k^2 y(t) = f(t)$

$$\mathcal{L}\{y''(t)\} + k^2 \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) + k^2 Y(s) = F(s)$$

$$Y(s)[s^2 + k^2] = F(s) - s\,y(0) - y'(0)$$

$$Y(s) = \frac{F(s) - s\,y(0) - y'(0)}{s^2 + k^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{F(s) - s\,y(0) - y'(0)}{s^2 + k^2}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} - c_1 \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} - \frac{c_2}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

where $y(0) = c_1$, $y'(0) = c_2$

$$y(t) = \frac{1}{k} f(t) * \sin kt - c_1 \cos kt - \frac{c_2}{k} \sin kt$$

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b) Given

$$y''(t) - 2k\,y'(t) + k^2 y(t) = f(t)$$

Applying L.T.

$$s^2 Y(s) - s y(0) - y'(0) - 2k\,sY(s) + 2k\,y(0) + k^2 Y(s) = F(s)$$

$$(s^2 - 2ks + k^2)Y(s) = F(s) + s y(0) + y'(0) - 2k\,y(0)$$

$$Y(s) = \frac{F(s) + s y(0) + y'(0) - 2k\,y(0)}{(s-k)^2}$$

Applying I.L.T. $y(t) = \mathcal{L}^{-1}\left\{\dfrac{F(s) + s y(0) + y'(0) - 2k\,y(0)}{(s-k)^2}\right\}$

$$y(t) = f(t) * te^{kt} + c_1(e^{kt} + kte^{kt}) + c_2 te^{kt} - 2kc_1 te^{kt}$$

c) Given: $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$

Applying L.T. on both sides

$$s^2 Y(s) - s y(0) - y'(0) + \lambda s Y(s) - \lambda y(0) + k^2 Y(s) = F(s)$$

$$(s^2 + \lambda s + k^2)Y(s) = F(s) + s y(0) + y'(0) + \lambda y(0)$$

$$Y(s) = \frac{F(s) + s y(0) + y'(0) + \lambda y(0)}{s^2 + \lambda s + k^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{F(s) + s y(0) + y'(0) + \lambda y(0)}{s^2 + \lambda s + k^2}\right\}$$

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Q. 10. Solve the problem

$y'' + \omega^2 y = f(t)$ $\quad y(0) = y_0, \quad y'(0) = y_1$

and discuss the case when

$$f(t) = \begin{cases} f_0 & , t_0 < t < t_1 \\ 0 & \text{for other values of } t \end{cases}$$

Sol.

Given

$y'' + \omega^2 y = f(t)$

$\mathcal{L}\{y''\} + \omega^2 \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$

$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + \omega^2 Y(s) = F(s)$

$\Rightarrow (s^2 + \omega^2) Y(s) = F(s) + s y_0 + y_1$

$\Rightarrow Y(s) = \dfrac{F(s)}{s^2+\omega^2} + \dfrac{s}{s^2+\omega^2} \cdot y_0 + \dfrac{1}{s^2+\omega^2} \cdot y_1$

$y(t) = y_0 \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+\omega^2}\right\} + \dfrac{y_1}{\omega} \mathcal{L}^{-1}\left\{\dfrac{\omega}{s^2+\omega^2}\right\}$

$+ \dfrac{1}{\omega}\mathcal{L}^{-1}\left\{F(s) \cdot \dfrac{\omega}{s^2+\omega^2}\right\}$

$\Rightarrow y(t) = y_0 \cos \omega t + \dfrac{y_1}{\omega} \sin \omega t + \dfrac{1}{\omega} f(t) * \sin \omega t$

Now $f(t) * \sin \omega t = \int_0^t f(\xi) \sin \omega(t-\xi)\, d\xi$

$= \int_0^{t_0} f(\xi) \sin \omega(t-\xi)\, d\xi + \int_{t_0}^{t_1} f(\xi) \sin \omega(t-\xi)\, d\xi$

$+ \int_{t_1}^{t} f(\xi) \sin \omega(t-\xi)\, d\xi$

$= 0 + \int_{t_0}^{t_1} f(\xi) \sin \omega(t-\xi)\, d\xi + 0$

$= \int_{t_0}^{t_1} f(\xi) \sin \omega(t-\xi)\, d\xi$

$\Rightarrow y(t) = y_0 \cos \omega t + \dfrac{y_1}{\omega} \sin \omega t + \dfrac{1}{\omega} \int_{t_0}^{t_1} f(\xi) \sin \omega(t-$

Q.11:- Solve the inhomogeneous problems with zero initial conditions i.e.

$$u(0)=0 \quad \text{and} \quad u'(0)=0$$

a) $u''+au=1$
b) $u''+u=t$
c) $u''+2u'=1-e^{-t}$
d) $u''-u=1$
e) $u''+4u=\sin t$

Sol.

(a) $\mathcal{L}\{u''\}+\mathcal{L}\{au\}=\mathcal{L}\{1\}$

$$s^2U(s)-su(0)-u'(0)+aU(s)=1/s$$

$$(s^2+a)U(s)=1/s$$

$$U(s)=\frac{1}{s(s^2+a)}$$

$$\Rightarrow u(t)=\frac{1}{a}(1-\cos\sqrt{a}\,t) \quad \text{by exp 1}$$

b) $\mathcal{L}\{u''\}+\mathcal{L}\{u\}=\mathcal{L}\{t\}$

$$s^2U(s)-su(0)-u'(0)+U(s)=1/s^2$$

$$(s^2+1)U(s)=\frac{1}{s^2}$$

$$U(s)=\frac{1}{s^2(s^2+1)}$$

$$\therefore \mathcal{L}^{-1}\{U(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}=t*\sin t$$
$$=\sin t * t$$

$$u(t)=\int_0^t \sin\tau\,(t-\tau)\,d\tau$$

$$=(t-\tau)\cdot[-\cos\tau]\Big|_0^t-\int_0^t(0-1)(-\cos\tau)\,d\tau$$

$$=(t-t)[-\cos t]-(t-0)(-\cos(0))-\sin\tau\Big|_0^t$$

$$=t-\sin t$$

$$\Rightarrow u(t)=t-\sin t$$

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c)

$$\mathcal{L}\{u''\} + 2\mathcal{L}\{u'\} = \mathcal{L}\{1 - e^{-t}\}$$

$$s^2U(s) - su(0) - u'(0) + 2sU(s) - 2u(0) = \frac{1}{s} - \frac{1}{s+1}$$

$$(s^2+2)U(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$U(s) = \frac{1}{s(s^2+2)} - \frac{1}{(s+1)(s^2+2)}$$

$$u(t) = \frac{1}{2}(1 - \cos\sqrt{2}t) - \frac{1}{\sqrt{2}}e^{-t} * \sin\sqrt{2}t$$

$$= \frac{1}{2}(1 - \cos\sqrt{2}t) - \frac{1}{\sqrt{2}}\left[\int_0^t e^{-\tau}\sin\sqrt{2}(t-\tau)\,d\tau\right]$$

Let $I = \int_0^t e^{-\tau}\sin\sqrt{2}(t-\tau)\,d\tau$

$$= e^{-\tau}\left(\frac{+\cos\sqrt{2}(t-\tau)}{+\sqrt{2}}\right)\Big|_0^t - \int(-e^{-\tau})\left(\frac{+\cos\sqrt{2}(t-\tau)}{+\sqrt{2}}\right)d\tau$$

$$= e^{-t} - \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\left[e^{-\tau}\left(\frac{\sin\sqrt{2}(t-\tau)}{-\sqrt{2}}\right)\Big|_0^t - \int_0^t(e^{-\tau})\left(\frac{\sin\sqrt{2}(t-\tau)}{-\sqrt{2}}\right)d\tau\right]$$

$$= e^{-t} - \cos\sqrt{2}t + \frac{1}{2}\sin\sqrt{2}t - \frac{1}{2}[I]$$

$$\Rightarrow I = \frac{2}{3}(e^{-t} - \cos\sqrt{2}t) + \frac{1}{3}\sin\sqrt{2}t$$

$$\Rightarrow u(t) = \frac{1}{2}(1 - \cos\sqrt{2}t) - \frac{\sqrt{2}}{3}(e^{-t} - \cos\sqrt{2}t) - \frac{1}{2\sqrt{3}}\sin\sqrt{2}t$$

d)

$$u'' - u = 1 \Rightarrow s^2U(s) - su(0) - u'(0) - U(s) = \frac{1}{s}$$

$$U(s) = \frac{1}{s(s^2-1)} \Rightarrow u(t) = 1 * \sinh t = \sinh t * 1$$

$$= \int_0^t \sinh\tau\,(t-\tau)^0\,d\tau = \cosh\tau\Big|_0^t = \cosh t - 1$$

$$\Rightarrow u(t) = \cosh t - 1$$

$$\Rightarrow u(t) = \cosh t - 1$$

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e) Given

$$u'' + 4u = \sin t$$

$$s^2U(s) - s\,u(0) - u'(0) + 4U(s) = \frac{1}{s^2+1}$$

$$U(s) = \frac{1}{(s^2+4)(s^2+1)}$$

$$u(t) = \frac{1}{2}\sin 2t * \sin t$$

$$= \frac{1}{2}\int_0^t \sin 2\tau \,\sin(t-\tau)\, d\tau$$

$$= -\frac{1}{4}\int_0^t \left(-2\sin 2\tau \,\sin(t-\tau)\right) d\tau$$

$$= -\frac{1}{4}\int_0^t \left(\cos(\tau+t) - \cos(3\tau-t)\right) d\tau$$

$$= -\frac{1}{4}\sin(\tau+t)\Big|_0^t + \frac{1}{12}\sin(3\tau-t)\Big|_0^t$$

$$= -\frac{1}{4}(\sin 2t - \sin t) + \frac{1}{12}(\sin 2t + \sin t)$$

$$= -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t$$

4410622

Q.12. Find I.L.T. by Partial Fraction.

a) $\frac{1}{s^2-4}$ b) $\frac{s+3}{s(s^2+2)}$ c) $\frac{1}{s(s+1)}$

Sol a) Given $\frac{1}{s^2-4}$

$$\frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + B(s-2)$$

$s = 2 \Rightarrow 1 = 4A \Rightarrow$ $\boxed{A = 1/4}$

$s = -2 \Rightarrow 1 = -4B \Rightarrow$ $\boxed{B = -1/4}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

b) Let

$$\frac{s+3}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

$$s+3 = A(s^2+2) + s(Bs+C)$$

$$= (A+B)s^2 + Cs + 2A$$

Comparing coefficients $2A = 3 \Rightarrow A = 3/2$

$\boxed{C=1}$ and $A + B = 0 \Rightarrow B = -3/2$

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So $\mathcal{L}^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{3}{2}s+1}{s^2+2}\right\}$

$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$

$= \frac{3}{2}(1) - \frac{3}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = \frac{3}{2} - \frac{3}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$

c) $\frac{4}{s(s+1)} = 4\left[\frac{1}{s} - \frac{1}{s+1}\right]$

$\mathcal{L}^{-1}\left\{\frac{4}{s(s+1)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= 4(1) - 4e^{-t}$

$= 4(1-e^{-t})$

Q.13: Solve

$y''(t) + ty'(t) - y(t) = 0$

with $y(0) = 0$ $\quad y'(0) = 1$

Sol.

$\mathcal{L}\{y''(t)\} + \mathcal{L}\{t\,y'(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{0\}$

$s^2Y(s) - sy(0) - y'(0) + (-1)^1\frac{d}{ds}\{sY(s) - y(0)\}$

$- Y(s) = 0$

(Here we have used the formula

$\mathcal{L}\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\{F(s)\}$ which will discussed later)

Now

$s^2Y(s) - 0 - 1 - sY'(s) - (1)Y(s) + 0 - Y(s) = 0$

$(s^2-2)Y(s) - sY'(s) - 1 = 0$

$sY'(s) - (s^2-2)Y(s) = -1$

$Y'(s) - \frac{s^2-2}{s}Y(s) = -1/s$

I.F. $= e^{-\int\frac{s^2-2}{s}ds} = e^{-\int(s-\frac{2}{s})ds}$

$= e^{-s^2/2 + \ln s^2} = e^{-s^2/2}\cdot e^{\ln s^2} = s^2e^{-s^2/2}$

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xing both sides by $s^2 e^{-s^2/2}$ and integrating

$$\int d\left(s^2 e^{-s^2/2} Y(s)\right) = \int \left(-s e^{-s^2/2}\right) ds$$

$$s^2 e^{-s^2/2} Y(s) = e^{-s^2/2} + C$$

Now $y(0) = 0$

$$\mathcal{L}\{y(0)\} = \mathcal{L}\{0\}$$

$$Y(0) = 0$$

$$0 = e^0 + C \Rightarrow \boxed{C = -1}$$

$$\Rightarrow s^2 e^{-s^2/2} Y(s) = e^{-s^2/2} - 1$$

$$\Rightarrow Y(s) = \frac{1}{s^2} - \frac{e^{s^2/2}}{s^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{s^2/2}}{s^2}\right\}$$

$$\Rightarrow y(t) = t - \mathcal{L}^{-1}\left\{\frac{e^{s^2/2}}{s^2}\right\}$$

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Laplace Transform of Unit Step Function

$$\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt$$

$$= \int_0^a e^{-st} H(t-a)\,dt + \int_a^\infty e^{-st} H(t-a)\,dt$$

$$= \int_0^a e^{-st} (0) + \int_a^\infty e^{-st} (1)\,dt$$

$$= 0 + \frac{e^{-st}}{-s}\Big|_a^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-sa}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-sa}\right]$$

$$\Rightarrow \mathcal{L}\{H(t-a)\} = \frac{1}{s} e^{-sa}$$

Theorem:- Prove that

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Proof:- For $n=1$

$$\mathcal{L}\{t f(t)\} = \int_0^\infty e^{-st} (t f(t))\,dt$$

Now $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$F'(s) = \int_0^\infty (-t) e^{-st} f(t)\,dt = -\int_0^\infty e^{-st} (t f(t))\,dt$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = (-1)^1 \frac{d}{ds}\{F(s)\}$$

$\Rightarrow$ C-1 is satisfied.

suppose it is true for $n=k$ i.e. $\mathcal{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k}$

Now we prove it for $n=k+1$

$$(-1)^k \frac{d^k}{ds^k}\{F(s)\} = \mathcal{L}\{t^k f(t)\} = \int_0^\infty e^{-st} (t^k f(t))\,dt$$

Diff. w.r.t. s

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}\{F(s)\} = \int_0^\infty (-t e^{-st}) (t^k f(t))\,dt$$

$$\Rightarrow (-1)^k \frac{d^{k+1}}{ds^{k+1}}\{F(s)\} = -\int_0^\infty e^{-st} (t^{k+1} f(t))\,dt$$

$$\Rightarrow (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\{F(s)\} = \mathcal{L}\{t^{k+1} f(t)\}$$

$\Rightarrow$ Th. is true for $n=k+1$ $\Rightarrow$ C-2 is satisfied

So theorem is true for all derivatives upto nth order

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THE SECOND SHIFTING (TRANSLATION) THEOREM

Statement: It states that

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a) \; ; \; a>0$$

Proof: To prove

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

we prove $\mathcal{L}\{H(t-a)\,f(t-a)\} = e^{-as}F(s)$

$$\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_0^\infty H(t-a)\,f(t-a)\,e^{-st}\,dt$$

$$= \int_0^a H(t-a)\,f(t-a)\,e^{-st}\,dt + \int_a^\infty H(t-a)\,f(t-a)\,e^{-st}\,dt$$

$$= \int_0^a e^{-st}(0)\,f(t-a)\,dt + \int_a^\infty e^{-st}(1)\,f(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = \mathfrak{J}$

$t = a \Rightarrow \mathfrak{J} = 0$

$t = \infty \Rightarrow \mathfrak{J} = \infty$

$dt = d\mathfrak{J}$

So $\mathcal{L}\{H(t-a)\,f(t-a)\} = \int_0^\infty e^{-s(\mathfrak{J}+a)} f(\mathfrak{J})\,d\mathfrak{J}$

$$= e^{-as}\int_0^\infty e^{-s\mathfrak{J}} f(\mathfrak{J})\,d\mathfrak{J}$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-as}\,\mathcal{L}\{f(t)\} = e^{-as}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

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Corollary: Prove that

$$\mathcal{L}\{P_n(t)\, f(t)\} = P(-D)\, F(s)$$

Proof:

Let

$$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

$$\mathcal{L}\{P_n(t) f(t)\} = \mathcal{L}\{(a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) f(t)\}$$

$$= a_n \mathcal{L}\{t^n f(t)\} + a_{n-1} \mathcal{L}\{t^{n-1} f(t)\} + \cdots + a_1 \mathcal{L}\{t f(t)\} + a_0 \mathcal{L}\{f(t)\}$$

$$= a_n \{(-1)^n D^n (F(s))\} + a_{n-1} \{(-1)^{n-1} D^{n-1} (F(s))\} + \cdots + a_1 \{(-1)^1 D(F(s))\} + a_0 F(s)$$

$$= [a_n (-D)^n + a_{n-1} (-D)^{n-1} + \cdots + a_1 (-D)^1 + a_0]\, F(s)$$

$$= P_n(-D)\, F(s)$$

$$\Rightarrow \mathcal{L}\{P_n(t)\, f(t)\} = P_n(-D)\, F(s)$$

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$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

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THE LAPLACE INVERSION INTEGRAL or FOURIER MELLIN INTEGRAL

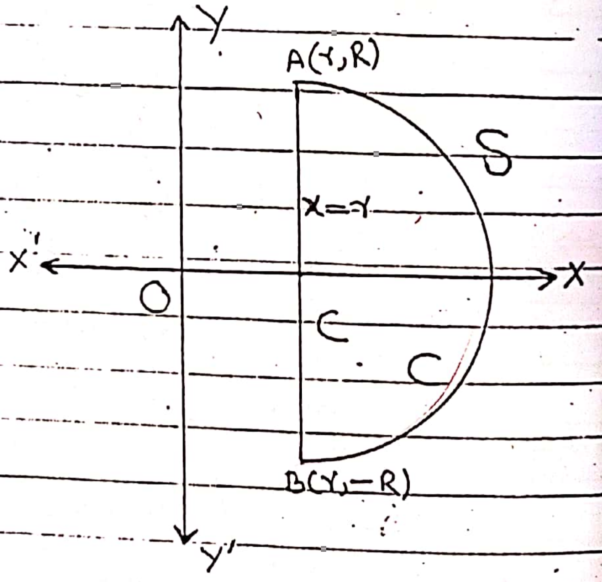
Statement:-

If $f(t)$ is the inverse Laplace Transform of $F(s)$, and all the singularities of $F(s)$ in the complex plane lie to the left of the line $x=\gamma$, then

$$f(t)=\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds$$

PROOF:-

Draw the line $x=\gamma$ in the XY-plane and mark the point $A(\gamma,R)$ and $B(\gamma,-R)$ on the line and draw a semicircle S of radius R to right of the line $x=\gamma$ as shown in the figure. Consider the closed contour C consisting of the line segment $\overline{AB}$ and the semicircle S i.e. $C=\overline{AB}\cup S$. Now consider the function $F(z)$, then $F(z)$ is analytic on and within the closed contour C because all the singularities of $F(z)$



lie to the left of the line $x=\gamma$. If s is any point inside the C then by Cauchy's Integral Theorem

$$F(s) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z-s}\,dz \longrightarrow (1)$$

Where $F(z)$ is L.T. of $f(t)$

i.e.

$$F(z) = \int_0^\infty e^{-zt} f(t)\,dt \longrightarrow (2)$$

From (1) and (2)

$$F(s) = \frac{1}{2\pi i}\int_C \frac{1}{z-s}\left(\int_0^\infty e^{-zt} f(t)\,dt\right)dz$$

since z and t are independent. So by interchanging the integrals we get

$$F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left(\int_C \frac{e^{-zt}}{z-s}\,dz\right)dt$$

$$\Rightarrow F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left(\int_S \frac{e^{-zt}}{z-s}\,dz + \int_A^B \frac{e^{-zt}}{z-s}\,dz\right)dt \longrightarrow (3)$$

Now by Jordan's Lemma

$$\int_S \frac{e^{-zt}}{z-s}\,dz = 0 \quad \text{When } R \to \infty$$

and hence

$$\int_A^B \frac{e^{-zt}}{z-s}\,dz = \int_{\gamma+i\infty}^{\gamma-i\infty} \frac{e^{-zt}}{z-s}\,dz$$

So by (3)

$$F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{s-z}\,dz\,dt$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{s-z}\left(\int_0^\infty f(t)\cdot e^{-zt}\,dt\right)dz$$

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$$F(s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(z)\cdot\frac{1}{s-z}\,dz$$

$$L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(z)\, L^{-1}\left\{\frac{1}{s-z}\right\} dz$$

$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(z)\, e^{zt}\,dz$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds$$

REMARK:-

By L.I.I. Formula

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{C} e^{st} F(s)\,ds$$

$$= \frac{1}{2\pi i}\left(\sum_j 2\pi i\, R_j\right)$$

$$\Rightarrow f(t) = \sum_j R_j$$

Where R_j = residue of $e^{st} F(s)$ at the Pole $s = s_j$.

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Illustrative Examples.

Exp. 1: Use Inversion Integral to evaluate $\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$

Sol. Here

$$F(s) = \frac{s^3+2s^2+1}{s^2(s^2+1)}$$

$F(s)$ is simple pole at $s=\pm i$ and double pole at $s=0$

So

$$R_1 = \lim_{s\to i}(s-i)\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}e^{st}$$

$$= \frac{-i-2+1}{-1(2i)}e^{it} = \frac{i+1}{2i} = \frac{-i(i+1)}{2}e^{it}$$

$$= \frac{1-i}{2}e^{it}$$

$$R_2 = \lim_{s\to -i}(s+i)\frac{s^3+2s^2+1}{s^2(s+i)(s-i)}e^{st}$$

$$= \frac{i-2+1}{-1(-2i)}e^{-it} = \frac{i-1}{2i}e^{-it} = -i\left(\frac{i-1}{2}\right)e^{-it}$$

$$= \frac{1}{2}(1+i)e^{-it}$$

$$R_3 = \lim_{s\to 0}\frac{d}{ds}\left\{(s-0)^2\frac{s^3+2s^2+1}{s^2(s^2+1)}e^{st}\right\}$$

$$= \lim_{s\to 0}\left\{te^{st}\frac{s^3+2s^2+1}{s^2+1} + e^{st}\frac{(s^2+1)(3s^2+4s)-(s^3+2s^2+1)(2s)}{(s^2+1)^2}\right\}$$

$$= t(1)+0 = t$$

So by Inversion Integral Formula

$$f(t) = R_1+R_2+R_3 = \frac{1}{2}(1-i)e^{it} + \frac{1}{2}(1+i)e^{-it} + t$$

$$= \frac{1}{2}\left[(1-i)(\cos t + i\sin t)\right] + \frac{1}{2}\left[(1+i)(\cos t - i\sin t)\right] + t$$

$$= \frac{1}{2}\left[\cos t + i\sin t - i\cos t + \sin t + \cos t - i\sin t + i\cos t + \sin t\right] + t$$

$$= \cos t + \sin t + t$$

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Exp. 2:- Calculate by Inversion Integral Method (a) $L^{-1}\left\{\frac{2(s+1)}{s(s^2+1)}\right\}$

(b) $L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

Sol.

a) $\frac{2(s+1)}{s(s^2+1)}$ has all simple poles which are $0, -i$, and $+i$. Residues at these be respectively R_0, R_1 and R_2

$R_0 = \lim_{s\to 0}\left\{(s-0)\frac{2(s+1)}{s(s^2+1)}\cdot e^{st}\right\} = \frac{2(0+1)}{0+1}e^0 = 2$

$\Rightarrow R_0 = 2$

$R_1 = \lim_{s\to -i}\left\{(s+i)\frac{2(s+1)}{s(s+i)(s-i)}e^{st}\right\} = \frac{2(1-i)}{-i(-2i)}e^{-it}$
$= (i-1)e^{-it}$

$= (i-1)(\cos t - i\sin t) = i\cos t - \cos t + \sin t + i\sin t$

$R_2 = \lim_{s\to i}\left\{(s-i)\cdot\frac{2(s+1)}{s(s-i)(s+i)}e^{st}\right\}$

$= \frac{2(i+1)}{i(2i)}e^{it} = -(1+i)(\cos t + i\sin t)$
$= -\cos t - i\cos t - i\sin t + \sin t$

Now by Inversion Integral Formula

$f(t) = R_0 + R_1 + R_2$

$= 2 + 2\sin t - 2\cos t$

$\Rightarrow f(t) = 2(\sin t - \cos t + 1)$

b) $\frac{1}{s^2(s+1)}$ has simple pole at $s=-1$ and double pole at $s=0$

$R_0 =$ Residue at $s=0 = \lim_{s\to 0}\frac{d}{ds}\left\{(s-0)^2\cdot\frac{1}{s^2(s+1)}\cdot e^{st}\right\}$

$= \lim_{s\to 0}\left\{te^{st}/_{s+1} - e^{st}\cdot\frac{1}{(s+1)^2}\right\}$

$= t-1$

$R_1 =$ Res. at $s=-1 = \lim_{s\to -1}\left\{(s+1)\cdot e^{st}/s^2(s+1)\right\} = \frac{e^{-t}}{1} = e^{-t}$

So $f(t) = R_0 + R_1 = t - 1 + e^{-t}$

$\Rightarrow f(t) = e^{-t} + t - 1$

Applications to P.D.Eqs.

Before studying L.T. to P.D.Eqs the following results must be kept in view.

1) $\mathcal{L}\{u(x,t)\} = U(x,s)$

2) $\mathcal{L}\{\frac{\partial u}{\partial x}\}$

$$= \int_0^\infty e^{-st} \frac{\partial u}{\partial x} dt$$

$$= \frac{\partial}{\partial x} \int_0^\infty e^{-st} u(x,t) dt$$

$$= \frac{\partial}{\partial x} \{\mathcal{L}\{u(x,t)\}\} = \frac{\partial}{\partial x} U(x,s)$$

3) $\mathcal{L}\{\frac{\partial u}{\partial t}\}$

$$= s\,\mathcal{L}\{u\} - u(x,0)$$

$$= sU(x,s) - u(x,0)$$

4) $\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\}$

$$= \frac{\partial^2}{\partial x^2} U(x,s)$$

5) $\mathcal{L}\{\frac{\partial^2 u}{\partial t^2}\}$

$$= s^2 U(x,s) - s u(x,0) - u'(x,0)$$

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Illustrative Examples.

Exp. 1:-

Use L.T. Method to solve the problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ $\quad 0<x<a$, $0<t<\infty$

with $u(0,t)=0$ $\quad u(1,t)=0$ $\quad t>0$

$u(x,0) = 1+\sin\pi x$ $\quad 0<x<a$

Sol.

It is a one dimensional heat problem describing conduction of heat through a rod of unit length, whose end-points are maintained at zero temperature, and whose initial temperature profile is prescribed.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$\Rightarrow L\{\partial^2 u/\partial x^2\} = L\{\partial u/\partial t\}$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = sU(x,s) - u(x,0)$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = sU(x,s) - 1 - \sin\pi x$

$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - sU(x,s) = -(1+\sin\pi x)$

A.E. is $\quad D^2 - s = 0$

$\Rightarrow D = \pm\sqrt{s}$

$\Rightarrow U_c = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

Now

$U_p = \frac{1}{D^2 - s}\left[-(1+\sin\pi x)\right]$

$= -\frac{1}{D^2-s} e^{0x} + \text{Im}\, \frac{1}{D^2-s} e^{i\pi x}$

$= \frac{-1}{0^2-s} - \text{Im}\left[\frac{e^{i\pi x}}{-\pi^2 - s}\right]$

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$$= \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s^2}$$

Therefore

$$U(x,s) = U_c + U_p$$
$$= c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}$$

$$U(0,s) = c_1 + c_2 + \frac{1}{s}$$

Also $u(0,t) = 0$

$\Rightarrow U(0,s) = 0$

$\Rightarrow c_1 + c_2 + \frac{1}{s} = 0$

$$U(1,s) = c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s}$$

$u(1,t) = 0$

$\Rightarrow U(1,s) = 0$

$\Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} = 0$

$\Rightarrow c_1 = 0, \; c_2 = 0$

So

$$U(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}$$

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{\sin \pi x}{s + \pi^2}\right\}$$

$$\Rightarrow u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x$$

$$\Rightarrow u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x$$

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Exp. 2:-

Solve the problem by L.T. Method

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) \qquad , t > 0, x > 0$$

$$u(x,0) = 0, \; u_t(x,0) = 0$$

$$u(0,t) = f(t)$$

and $\lim_{x \to \infty} u(x,t) = 0$

Sol. Given $u_{tt}(x,t) = a^2 u_{xx}(x,t)$

$$\mathcal{L}\{u_{tt}(x,t)\} = a^2 \mathcal{L}\{u_{xx}(x,t)\}$$

$$\Rightarrow s^2 U(x,s) - s\,u(x,0) - u_t(x,0) = a^2 \frac{\partial^2}{\partial x^2} U(x,s)$$

$$\Rightarrow s^2 U(x,s) - 0 - 0 = a^2 \frac{\partial^2}{\partial x^2} U(x,s)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{s^2}{a^2}\right) U(x,s) = 0$$

A.E. is $D^2 - s^2/a^2 = 0$

$\Rightarrow D = \pm s/a$

$\Rightarrow U(x,s) = c_1 e^{sx/a} + c_2 e^{-sx/a}$

Where c_1, c_2 may or may not depend upon s but it is independent of x.

$U(0,s) = c_1 + c_2$

As $u(0,t) = f(t) \Rightarrow U(0,s) = F(s)$

So $c_1 + c_2 = F(s) \longrightarrow$ ①

Also as

$$\lim_{x\to\infty} u(x,t) = 0$$

$$\Rightarrow \lim_{x\to\infty} U(x,s) = 0$$

$$\Rightarrow \lim_{x\to\infty}\left(c_1 e^{sx/a} + c_2 e^{-sx/a}\right) = 0$$

$$\Rightarrow c_1 = 0$$

Then by ① $c_2 = F(s)$

$$\Rightarrow U(x,s) = 0 + F(s)\, e^{-sx/a}$$

$$\Rightarrow U(x,s) = F(s)\, e^{-sx/a}$$

Applying $\mathcal{L}^{-1}$ on both sides

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\{e^{-sx/a} F(s)\}$$

$$\Rightarrow u(x,t) = H\left(t - \frac{x}{a}\right) f\left(t - \frac{x}{a}\right)$$

$$\Rightarrow u(x,t) = \begin{cases} f(t - x/a) & t > x/a \\ 0 & \end{cases}$$

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Example 3.

Use L.T. Method to Solve

$$U_{tt}(x,t) = a^2 U_{xx}(x,t) - g$$

$$u(x,0) = u_t(x,0) = 0$$

$$u(0,t) = 0$$

$$\lim_{x\to\infty} U_x(x,t) = 0$$

Sol.

$$\mathcal{L}\{U_{tt}(x,t)\} = a^2 \mathcal{L}\{U_{xx}(x,t)\} - \mathcal{L}\{g\}$$

$$\Rightarrow s^2 U(x,s) - s\,u(x,0) - u_t(x,0) = a^2 \frac{\partial^2}{\partial x^2} U(x,s) - g/s$$

$$\Rightarrow s^2 U(x,s) - 0 + 0 = a^2 \frac{\partial^2}{\partial x^2} U(x,s) - g/s$$

$$\left(a^2 \frac{\partial^2}{\partial x^2} - s^2\right) U(x,s) = g/s$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{s^2}{a^2}\right) U(x,s) = g/sa^2$$

A.E. is $D^2 - s^2/a^2 = 0 \Rightarrow D = \pm s/a$

So $U_c = c_1 e^{sx/a} + c_2 e^{-sx/a}$

$$U_p = \frac{1}{D^2 - \frac{s^2}{a^2}} \cdot g/sa^2$$

$$= \frac{g}{sa^2} \cdot \frac{1}{D^2 - \frac{s^2}{a^2}} e^{0x}$$

$$= \frac{g}{sa^2} \cdot \frac{1}{0 - \frac{s^2}{a^2}} = \frac{g}{sa^2} \cdot \left(-\frac{a^2}{s^2}\right) = -\frac{g}{s^3}$$

So $U(x,s) = U_c + U_p$

$$U(x,s) = c_1 e^{sx/a} + c_2 e^{-sx/a} - g s^{-3}$$

$$U(0,s) = c_1 + c_2 - g s^{-3} \quad \rightarrow (1)$$

$$\Rightarrow c_1 + c_2 - g s^{-3} = 0 \quad \leftarrow \because u(0,t) = 0 \Rightarrow U(0,s) = 0$$

Now $U_x(x,s) = \frac{s}{a} c_1 e^{sx/a} - \frac{s}{a} c_2 e^{-sx/a}$

$$\lim_{x\to\infty} U_x(x,s) = 0$$

$$\Rightarrow c_1 = 0$$

~~$\Rightarrow c_1 \ldots$~~

So by (1) $\quad \ldots + c_2 - g s^{-3} = 0$

$\Rightarrow c_2 = g s^{-3}$

$\Rightarrow c_1 = \frac{1}{2} g s^{-3}$

$\Rightarrow U(x,s) = (e^{sx/a} + e^{-sx/a} - 2)\frac{1}{2} g s^{-3}$

$$\Rightarrow U(x,s) = \frac{g}{s^3} e^{-sx/a} - g/s^3$$

$$\Rightarrow u(x,t) = g \mathcal{L}^{-1}\left\{e^{-sx/a} \cdot \frac{1}{s^3}\right\} - g \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{g}{2!} \mathcal{L}^{-1}\left\{e^{-\frac{sx}{a}} \cdot \frac{2!}{t^3}\right\} - \frac{g}{2!} \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$\Rightarrow u(x,t) = \frac{g}{2!} H\left(t - \frac{x}{a}\right)\left(t - \frac{x}{a}\right)^2 - \frac{g}{2!} t^2$$

$$\Rightarrow u(x,t) = \frac{g}{2} H\left(t - \frac{x}{a}\right)\left(t - \frac{x}{a}\right)^2 - \frac{g}{2} t^2$$

مثال ۲/۲: L.T طریقہ سے حل کریں

$$u_{xx} = u_{tt} \qquad 0 < x < 1, \quad t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0$$

$$u(x,0) = \sin \pi x$$

$$u_t(x,0) = -\sin \pi x$$

Sol.

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$$

$$\frac{\partial^2}{\partial x^2} U(x,s) = s^2 U(x,s) - s u(x,0) - u_t(x,0)$$

$$= s^2 U(x,s) - s \sin \pi x + \sin \pi x$$

$$\left(\frac{\partial^2}{\partial x^2} - s^2\right) U(x,s) = \sin \pi x - s \sin \pi x$$

A.E. is $\quad D^2 - s^2 = 0 \quad \Rightarrow D = \pm s$

$$\Rightarrow U_c = c_1 e^{sx} + c_2 e^{-sx}$$

$$U_p = \frac{1}{D^2 - s^2} \sin \pi x = \operatorname{Im} \frac{(1-s)}{D^2 - s^2} e^{i\pi x}$$

$$= \operatorname{Im} \frac{(1-s)}{(i\pi)^2 - s^2} e^{i\pi x} = \frac{-(1-s)\sin \pi x}{\pi^2 + s^2}$$

$$\Rightarrow U(x,s) = U_c + U_p = c_1 e^{sx} + c_2 e^{-sx} - \frac{(1-s)\sin \pi x}{s^2 + \pi^2}$$

$$U(1,s) = c_1 e^{s} + c_2 e^{-s} = 0 \qquad \because u(1,t) = 0$$

4410622

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$\Rightarrow c_1 e^{s} + c_2 e^{-s} = 0 \rightarrow (1)$

Also $u(0,t) = 0 \Rightarrow U(0,s) = 0$

$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

by (1) $\Rightarrow -c_2 e^{s} + c_2 e^{-s} = 0$

$c_2 (e^{-s} - e^{s}) = 0 \Rightarrow c_2 = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow U(x,s) = 0 + 0 + \frac{(1-s)\sin \pi x}{s^2 + \pi^2}$

$u(x,t) = \sin \pi x \left[\frac{1}{\pi} \cdot \mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}\right]$

$u(x,t) = \sin \pi x \left[\cos \pi t - \frac{1}{\pi} \sin \pi t \right]$

Exercise

Q1 is Exp. 3 and Q.2 is Exp. 4.

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Theorem.

If $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$

Proof:-

$$\int_s^{\infty} F(s)\,ds = \int_s^{\infty}\left(\int_0^{\infty} e^{-st} f(t)\,dt\right) ds$$

$$= \int_0^{\infty} f(t)\left(\int_s^{\infty} e^{-st}\,ds\right) dt \qquad \text{By interchanging integrals}$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\Big|_s^{\infty}\right] dt$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^{\infty} f(t)\left[0 + \frac{e^{-st}}{t}\right] dt = \int_0^{\infty} e^{-st}\left\{\frac{f(t)}{t}\right\} dt$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

$$\Rightarrow \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$$

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Examples.

Exp. 1:- Prove that $\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$

Sol.

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \mathcal{L}\left\{\frac{\cos at}{t}\right\} - \mathcal{L}\left\{\frac{\cos bt}{t}\right\}$$

$$= \int_s^{\infty}\frac{s}{s^2+a^2}\,ds - \int_s^{\infty}\frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\left[\ln|s^2+a^2| - \ln|s^2+b^2|\right]\Big|_s^{\infty}$$

$$= \frac{1}{2}\ln\frac{s^2+a^2}{s^2+b^2}\Big|_s^{\infty} = \frac{1}{2}\ln\left|\frac{1+a^2/s^2}{1+b^2/s^2}\right|\Big|_s^{\infty}$$

$$= \frac{1}{2}\ln(1) - \frac{1}{2}\ln\frac{1+a^2/s^2}{1+b^2/s^2} = -\frac{1}{2}\ln(s^2+a^2) + \frac{1}{2}\ln(s^2+b^2)$$

$$= \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$$

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Ex. 2: Evaluate

$$\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}$$

Sol.

Let $f(t) = \frac{1-\cos t}{t}$

Then $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\left\{\frac{1-\cos t}{t}\right\}$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \frac{1}{2}\ln s^2 - \frac{1}{2}\ln|s^2+1| \Big|_s^\infty$$

$$= \frac{1}{2}\ln(s^2+1) - \frac{1}{2}\ln(s^2)$$

$$= \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

Now

$$\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\} = \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\left[\int_s^\infty 1\cdot\ln(s^2+1)\,ds - 2\int_s^\infty 1\cdot\ln s\,ds\right]$$

$$= \frac{1}{2}\left[\ln(s^2+1)\cdot s\Big|_s^\infty - \int_s^\infty \frac{2s}{s^2+1}\cdot s\,ds - 2\left\{\ln s\cdot s\Big|_s^\infty - \int_s^\infty \frac{1}{s}\cdot s\,ds\right\}\right]$$

$$= \frac{1}{2}\left[s\ln(s^2+1)\Big|_s^\infty - 2\int_s^\infty \left(1 - \frac{1}{s^2+1}\right) ds - 2s\ln s\Big|_s^\infty + 2s\Big|_s^\infty\right]$$

$$= \frac{1}{2}\left[s\ln(s^2+1)\Big|_s^\infty - 2s\Big|_s^\infty + 2\tan^{-1}s\Big|_s^\infty - 2s\ln s\Big|_s^\infty + 2s\Big|_s^\infty\right]$$

$$= \frac{1}{2}\left[s(\ln(s^2+1) - \ln s^2)\Big|_s^\infty + 2\left[\tan^{-1}(\infty) - \tan^{-1}(s)\right]\right]$$

$$= \frac{1}{2}\left[s(\ln s^2 - \ln(s^2+1)) + 2\left(\frac{\pi}{2} - \tan^{-1}s\right)\right]$$

$$= \frac{s}{2}\ln\frac{s^2}{s^2+1} + \frac{\pi}{2} - \tan^{-1}s$$

$$= \frac{s}{2}\ln\frac{s^2}{s^2+1} + \cot^{-1}s$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\} = \frac{s}{2}\ln\frac{s^2}{s^2+1} + \cot^{-1}s$$

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Formula:

$$F(s) = -\int_s^{\infty} F'(s)\,ds = -\mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

$$-\int_s^{\infty} F'(s)\,ds = -F(s)\Big|_s^{\infty} = -F(\infty) + F(s) = 0 + F(s) = F(s)$$ (∵ see page no. 113)

Where $f(t) = \mathcal{L}^{-1}\{F'(s)\}$

Exp. 1: Calculate

$$\mathcal{L}^{-1}\{\tan^{-1} a/s\}$$

Sol.

$$\tan^{-1} a/s = -\int_s^{\infty} (\tan^{-1} a/s)'\,ds$$

$$= -\int_s^{\infty} \frac{1}{\frac{a^2+s^2}{s^2}} \cdot \left(-\frac{a}{s^2}\right) ds$$

$$= +\int_s^{\infty} \frac{a}{s^2+a^2}\,ds$$

$$= \mathcal{L}\left\{\frac{\sin at}{t}\right\} \quad \because \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$$

$$\Rightarrow \mathcal{L}^{-1}\{\tan^{-1} a/s\} = \frac{1}{t}\sin at$$

Exp. 2: Calculate $\mathcal{L}^{-1}\left\{\ln \frac{s^2+1}{(s-1)^2}\right\}$.

Sol.

$$\ln \frac{s^2+1}{(s-1)^2} = -\int_s^{\infty} \left(\ln \frac{s^2+1}{(s-1)^2}\right)' ds$$

$$= -\int_s^{\infty} \frac{(s-1)^2}{s^2+1} \cdot \left(\frac{s^2+1}{(s-1)^2}\right)' ds$$

$$= -\int_s^{\infty} \frac{(s-1)^2}{s^2+1} \cdot \frac{(s-1)^2(2s) - (s^2+1)(2)(s-1)}{(s-1)^4}\,ds$$

$$= -\int_s^{\infty} \frac{\cancel{(s-1)^2}}{s^2+1} \cdot \frac{\cancel{2s^3} - 4s^2 + \cancel{2s} - \cancel{2s^3} + 2s^2 - \cancel{2s} + 2}{\cancel{(s-1)^2}\,(s-1)^2}\,ds$$

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$$= -\int_s^\infty \frac{-2(s^2+1)}{(s^2+1)(s-1)^2}\,ds$$

$$= +2\int_s^\infty \frac{1}{(s-1)^2}\,ds$$

$$= -\int_s^\infty \frac{-2(s^2+1)}{(s^2+1)(s-1)^2}\,ds$$

$$= 2\int_s^\infty \frac{(s-1)(s+1)}{(s^2+1)(s-1)^2}\,ds$$

$$= 2\int_s^\infty \frac{s+1}{(s^2+1)(s-1)}\,ds$$

$$= 2\int_s^\infty \left[\frac{1}{s-1} - \frac{s}{s^2+1}\right]ds$$

$$= \mathcal{L}\{(2e^t - 2\cos t)/t\}$$

$$= \mathcal{L}\left\{\frac{2}{t}\{e^t - \cos t\}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{(s-1)^2}\right\} = \frac{2e^t}{t} - \frac{2\cos t}{t}$$

Exp. 3. Calculate $\mathcal{L}^{-1}\left\{\ln\frac{s^2+a^2}{s^2+b^2}\right\}$

Sol.

$$\ln\frac{s^2+a^2}{s^2+b^2} = -\int_s^\infty \left[\ln\frac{s^2+a^2}{s^2+b^2}\right]' ds$$

$$= -\int_s^\infty \left[\ln(s^2+a^2) - \ln(s^2+b^2)\right]' ds$$

$$= -\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= 2\mathcal{L}\left\{\frac{\cos bt}{t}\right\} - 2\mathcal{L}\left\{\frac{\cos at}{t}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\ln\frac{s^2+a^2}{s^2+b^2}\right\} = \frac{2\cos bt - 2\cos at}{t}$$

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LAPLACE TRANSFORM OF PERIODIC FUNCTION

Theorem:-

If $f(t)$ has period $T > 0$, then calculate $\mathcal{L}\{f(t)\}$

Proof:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Consider $\int_T^{2T} e^{-st} f(t)\,dt$. Put $t = \lambda + T$

$dt = d\lambda$

$t = T \Rightarrow \lambda = 0 \qquad t = 2T \Rightarrow \lambda = T$

So $\int_T^{2T} e^{-st} f(t)\,dt = \int_0^T e^{-s(\lambda+T)} f(\lambda+T)\,d\lambda$

$$= \int_0^T e^{-s\lambda} \cdot e^{-sT} f(\lambda)\,d\lambda \qquad \because f(\lambda+T) = f(\lambda)$$

$$= e^{-sT} \int_0^T e^{-s\lambda} f(\lambda)\,d\lambda$$

Now consider $\int_{2T}^{3T} e^{-st} f(t)\,dt$

put $t = 2T + \lambda$ then $\int_{2T}^{3T} e^{-st} f(t)\,dt$

$$= \int_0^T e^{-s(2T+\lambda)} f(\lambda + 2T)\,d\lambda = e^{-2sT} \int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

So $\mathcal{L}\{f(t)\} = \int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + e^{-sT} \int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + e^{-2sT} \int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + \cdots$

$$= \{1 + e^{-sT} + e^{-2sT} + \cdots\} \int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

$$= \{1 - e^{-sT}\}^{-1} \int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^T \frac{e^{-st} f(t)}{1 - e^{-sT}}\,dt$$

Q. Prove that $\mathcal{L}^{-1}\left\{\frac{A}{(s-k)^n}\right\} = \frac{At^{n-1}}{(n-1)!}e^{kt}$

Sol. $\mathcal{L}^{-1}\left\{\frac{A}{(s-k)^n}\right\}$

$= \mathcal{L}^{-1}\{F(s-k)\}$ By 1st shifting theorem

$= e^{kt}\mathcal{L}^{-1}\{F(s)\}$

$= e^{kt}\mathcal{L}^{-1}\left\{\frac{A}{s^n}\right\}$

$= \frac{Ae^{kt}}{(n-1)!}\mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\}$

$= \frac{Ae^{kt}}{(n-1)!}t^{n-1}$

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Applications to Miscellaneous Equations.

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Exp. Solve

$$\frac{dx}{dt}+\frac{dy}{dt}-4y=1$$

and $x+\frac{dy}{dt}-3y=t^2$, $x(0)=0$, $y(0)=0$

Sol.:-

Given $\dot{x}+\dot{y}-4y=1 \longrightarrow (1)$

and $x+\dot{y}-3y=t^2 \longrightarrow (2)$

From (1)

$\mathcal{L}\{\dot{x}\}+\mathcal{L}\{\dot{y}\}-4\mathcal{L}\{y\}=\mathcal{L}\{1\}$

$sX(s)-x(0)+sY(s)-y(0)-4Y(s)=1/s$

$sX(s)-0+sY(s)-0-4Y(s)=1/s$

$sX(s)+(s-4)Y(s)=1/s \longrightarrow (3)$

From (2)

$\mathcal{L}\{x\}+\mathcal{L}\{\dot{y}\}-3\mathcal{L}\{y\}=\mathcal{L}\{t^2\}$

$X(s)+sY(s)-y(0)-3Y(s)=2!/s^3$

$X(s)+(s-3)Y(s)=2/s^3 \longrightarrow (4)$

(3) $-$ s (4) gives

$sX(s)+(s-4)Y(s)=1/s$

$-sX(s)-s(s-3)Y(s)=-2/s^2$

$(s-4-s^2+3s)Y(s)=\frac{1}{s}-\frac{2}{s^2}$

$(s^2-4s+4)Y(s)=\frac{2}{s^2}-\frac{1}{s}$

$$Y(s)=\frac{2}{s^2(s-2)^2}-\frac{1}{s(s-2)^2}=\frac{2-s}{s^2(s-2)^2}$$

$$=\frac{1/4}{s}+\frac{1/2}{s^2}+\frac{-1/4}{s-2}$$

$y(t)=\frac{1}{4}+\frac{1}{2}t-\frac{1}{4}e^{2t} \longrightarrow *$

Now

$(s-3)(3)-(s-4)(4)$ gives

$[s(s-3)-(s-4)]X(s)=(s-3)\cdot\frac{1}{s}-(s-4)\cdot\frac{2}{s^3}$

$$(s^2-4s+4)X(s) = \frac{s^2(s-3)-2(s-4)}{s^3}$$

$$(s-2)^2 X(s) = \frac{s^3-3s^2-2s+8}{s^3}$$

$$X(s) = \frac{s^3-3s^2-2s+8}{s^3(s-2)^2}$$

$$= \frac{(s-2)(s^2-s-4)}{s^3(s-2)(s-2)}$$

$$\Rightarrow X(s) = \frac{s^2-s-4}{s^3(s-2)}$$

$$= \frac{1/4}{s} + \frac{3/2}{s^2} + \frac{2}{s^3} + \frac{-1/4}{s-2}$$

$$x(t) = \frac{1}{4} + \frac{3}{2}t + t^2 - \frac{1}{4}e^{2t} \rightarrow ** *$$

* and ** are resp. solutions.

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The Error Function

The error function of 'x' is denoted and defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\lambda^2}\,d\lambda$$

Inversion Formula:-

(Related to error function)

We have the following inversion formula related to error function

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = 1-\operatorname{erf}\left(\frac{2a}{\sqrt{t}}\right)$$

Which is equivalant to

$$\mathcal{L}\{\operatorname{erf} t^{-1/2}\} = \frac{1}{s}\{1-e^{-\sqrt{s}/2}\}$$

Q: Calculate

$$\mathcal{L}\{\operatorname{erf} t\}$$

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Sol.

$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\lambda^2}\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t \left[1+(-\lambda^2)+\frac{(-\lambda^2)^2}{2!}+\frac{(-\lambda^2)^3}{3!}+\cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t \left[1-\lambda^2+\frac{\lambda^4}{2}-\frac{\lambda^6}{6}+\cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[t-\frac{\lambda^3}{3}+\frac{\lambda^5}{10}-\frac{\lambda^7}{42}+\cdots\right]\Big|_0^t\,dt$$

$$= \frac{2}{\sqrt{\pi}}\left[t-\frac{t^3}{3}+\frac{t^5}{10}-\frac{t^7}{42}+\cdots\right]$$

$$\mathcal{L}\{\operatorname{erf} t\} = \frac{2}{\sqrt{\pi}}\left\{\frac{1}{s^2}-\frac{1}{3}\left(\frac{3!}{s^4}\right)+\frac{1}{10}\left(\frac{5!}{s^6}\right)-\frac{1}{42}\left(\frac{7!}{s^8}\right)+\cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{1}{s^2}-\frac{2}{s^4}+\frac{12}{s^6}-\frac{120}{s^8}+\cdots\right]$$

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Q. Calculate $\mathcal{L}\{e^{3t}\,\mathrm{erf}\sqrt{t}\}$

Sol.

$$\mathrm{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-\lambda^2}\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{e^{-t}}{-1}\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[-e^{-\sqrt{t}} + e^{0}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[1 - e^{-\sqrt{t}}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[1 - \left[1 + (-\sqrt{t}) + \frac{(-\sqrt{t})^2}{2!} + \frac{(-\sqrt{t})^3}{3!} + \frac{(-\sqrt{t})^4}{4!} + \cdots\right]\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{t} - \frac{t}{2} + \frac{t^{3/2}}{6} - \frac{t^2}{24} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-\lambda^2}\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - \lambda^2 + \frac{\lambda^4}{2} - \frac{\lambda^6}{6} + \frac{\lambda^8}{24} - \cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[\lambda - \frac{\lambda^3}{3} + \frac{\lambda^5}{10} - \frac{\lambda^7}{42} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{1}{3}t^{3/2} + \frac{1}{10}t^{5/2} - \frac{1}{42}t^{7/2} + \cdots\right]$$

$$\mathcal{L}\{e^{3t}\,\mathrm{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{10}\frac{\Gamma 7/2}{s^{7/2}} - \frac{1}{42}\frac{\Gamma 9/2}{s^{9/2}} + \cdots\right]\right]_{s\to s-3}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{3}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{10}\cdot\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}} - \frac{1}{42}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{9/2}} + \cdots\right]_{s\to s-3}$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}\sqrt{\pi}\left[\frac{1}{s^{3/2}} - \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{1}{s^{5/2}} + \frac{1}{10}\cdot\frac{15}{4}\cdot\frac{1}{s^{7/2}} - \frac{1}{42}\cdot\frac{105}{8}\cdot\frac{1}{s^{9/2}}\right]_{s\to s-3}$$

$$= \left[\frac{1}{s^{3/2}}\left(1 - \frac{1}{2s} + \frac{3}{8s^2} - \frac{5}{16s^3} + \cdots\right)\right]_{s\to s-3}$$

$$= \left[\frac{1}{s^{3/2}} \left[1 + \frac{1}{s} \right]^{-1/2} \right]_{s \to s-3} = \left[\frac{1}{s^{3/2}} \left\{ \frac{(s+1)^{-1/2}}{s^{-1/2}} \right\} \right]_{s \to s-3}$$

$$\Rightarrow \mathcal{L}\left\{ e^{3t} \operatorname{erf} \sqrt{t} \right\} = \frac{1}{s\sqrt{s+1}} \Bigg|_{s \to s-3}$$

$$\Rightarrow \mathcal{L}\left\{ e^{3t} \operatorname{erf} \sqrt{t} \right\} = \frac{1}{(s-3)\sqrt{s-2}}$$

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THE FOURIER TRANSFORMS AND ITS APPLICATIONS

DEFINITION:-

Fourier Transform of a function $f(x)$, if it exists, is denoted and defined by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

In this case $f(x)$ is called the inverse FOURIER TRANSFORM of $F(k)$ and it is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k)\,dk$$

DEFINITION:-

If $F(k)$ is the F.T. of a function $f(x)$ and $f(x)$ is the I.F.T. of $F(k)$ with

$$F(k) = c_1 \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

and $$f(x) = c_2 \int_{-\infty}^{\infty} e^{-ikx} F(k)\,dk$$

then c_1 and c_2 can be related as follows for each case

(i) $c_1 = 1/\sqrt{2\pi}$ $\quad c_2 = 1/\sqrt{2\pi}$

(ii) $c_1 = 1$ $\quad c_2 = 1/2\pi$

(iii) $c_1 = 1/2\pi$ $\quad c_2 = 1$

Fourier and its inverse transform can also be related by the following definitions

$$F(k) = \int_{-\infty}^{\infty} e^{2\pi i k x} f(x)\,dx$$

and

$$f(x) = \int_{-\infty}^{\infty} e^{-2\pi i k x} F(k)\,dk$$

Notations.

The pairs (x,k), (x,p), (x,ξ), (t,ω) are frequently also used by various authors. F.T. of $f(x)$ is also denoted by $F\{f(x)\}$ & $\bar{F}(k)$

Properties of F.T.

4410622

(i) Linearity Property:-

F.T. is Linear.

Proof:- $F\{\alpha_1 f_1(x) + \alpha_2 f_2(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(\alpha_1 f_1(x) + \alpha_2 f_2(x))\,dx$

$$= \alpha_1\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f_1(x)\,dx\right) + \alpha_2\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f_2(x)\,dx\right)$$

$$= \alpha_1 F\{f_1(x)\} + \alpha_2 F\{f_2(x)\}$$

(ii) Conjugation Property:-

If $f(x)$ is real then $F(-k) = \bar{F}(k)$

Proof:-

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$\Rightarrow \bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \quad \because f \text{ is real}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(-k)x} f(x)\,dx$$

$$\Rightarrow \bar{F}(k) = F(-k)$$

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(iii) Real & Complex Values of F.T.

a) If $f(x)$ is real and even then $F(k)$ is real

b) If $f(x)$ is real and odd then $F(k)$ is pure imaginary.

c) If $f(x)$ is complex, then

$$F\{\bar{f}(-x)\} = \bar{F}(k)$$

Proof:-

a)

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

$$\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx \qquad \because f \text{ is real}$$

Put $x = -x'$ $\quad dx = -dx'$

$$\Rightarrow \bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty} e^{-ik(-x')} f(-x')(-dx')$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')dx' \qquad \because f \text{ is even}$$

$$= F(k) \quad \Rightarrow \bar{F}(k) = F(k)$$

b)

$$\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{e^{ikx}} f(x)\,dx \qquad \because f \text{ is real}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

Put $x = -x'$ $\Rightarrow dx = -dx'$

So
$$\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty} e^{-ik(-x')} f(-x')(-dx')$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} [-f(x')](-dx') \qquad \because f \text{ is odd}$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')\,dx' = -F(k)$$

$$\Rightarrow \bar{F}(k) = -F(k)$$

$\Rightarrow F(k)$ is pure imaginary.

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c) $\bar{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \bar{f}(x)\, dx$

Put $x = -x'$ $\Rightarrow dx = -dx'$

So $\bar{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{-ik(-x')} \bar{f}(-x')(-dx')$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} \bar{f}(-x')\, dx'$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \bar{f}(-x)\, dx$

$\bar{F}(k) = F\{\bar{f}(-x)\}$

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(iv) Attenuation Property.

$F\{e^{ax} f(x)\} = F(k - ia)$

Proof:-

$F\{e^{ax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{ax} f(x) dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx + ax} f(x) dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k - ia)x} f(x) dx$

$\Rightarrow F\{e^{ax} f(x)\} = F(k - ia)$

(V) Shifting Property:-

$F\{f(x-a)\} = e^{ika} F(K)$

Proof:-

$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ikx} f(x-a)\, dx$

Put $x - a = t$ $dx = dt$

$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ik(t+a)} f(t)\, dt$

$= e^{ika} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikt} f(t) dt$

$= e^{ika} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$

$\Rightarrow F\{f(x-a)\} = e^{ika} F(k)$

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(vi) Scaling Property:-

If c is non zero constant then

$$F\{f(cx)\} = \frac{1}{|c|} F(k/c)$$

Proof:-

CASE:1:- If $c > 0$

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(cx)\,dx$$

Put $cx = x'$

$x = -\infty \Rightarrow x' = -\infty$

$x = \infty \Rightarrow x' = \infty$

and $c\,dx = dx' \Rightarrow dx = \frac{1}{c} dx'$

$$\Rightarrow F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x'/c)} f(x') \cdot \frac{1}{c} dx'$$

$$= \frac{1}{c}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k/c)x'} f(x')\,dx'\right]$$

$$= \frac{1}{c} F(k/c)$$

CASE:- If $c < 0$

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(cx)\,dx$$

Put $cx = x' \Rightarrow x = \frac{1}{c} x'$, $dx = \frac{1}{c} dx'$

$x = \infty \Rightarrow x' = -\infty$ $\because c < 0$

$x = -\infty \Rightarrow x' = \infty$ $\because c < 0$

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{ik(x'/c)} f(x') \frac{1}{c} dx'$$

$$= \frac{1}{-c}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k/c)x'} f(x')\,dx'\right]$$

$$= \frac{1}{-c} F(k/c)$$

Combining both cases $F\{f(cx)\} = \frac{1}{|c|} F(k/c)$

(vii) MODULATION PROPERTY OF F.T.

a) $F\{\cos\alpha x\, f(x)\} = \frac{1}{2}[F(k+\alpha) + F(k-\alpha)]$

b) $F\{\sin\alpha x\, f(x)\} = \frac{1}{2i}[F(k+\alpha) + F(k-\alpha)]$

PROOF:-

a) $F\{\cos\alpha x\, f(x)\}$

$= F\left\{\left(\frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\right) f(x)\right\}$

$= \frac{1}{2}\left[F\{e^{i\alpha x} f(x)\} + F\{e^{-i\alpha x} f(x)\}\right]$

$= \frac{1}{2}[F(k+\alpha) + F(k-\alpha)]$

b) $F\{\sin\alpha x\, f(x)\}$

$= F\left\{\left(\frac{e^{i\alpha x}-e^{-i\alpha x}}{2i}\right) f(x)\right\}$

$= \frac{1}{2i}\left[F\{e^{i\alpha x} f(x)\} - F\{e^{-i\alpha x} f(x)\}\right]$

$= \frac{1}{2i}[F(k+\alpha) - F(k-\alpha)]$

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Theorem:-

The F.T. $F(k)$ of $f(x)$ exists if f is absolutely integrable over $]-\infty, \infty[$

Proof:-

If f is absolutely integrable on $]-\infty, \infty[$, then $\int_{-\infty}^{\infty} |f(x)| dx$ exist.

Now $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ikx} f(x) dx$

Now $\int_{-\infty}^{\infty} |e^{ikx} f(x)| dx = \int_{-\infty}^{\infty} |e^{ikx}| |f(x)| dx$

$= \int_{-\infty}^{\infty} (1) |f(x)| dx \quad \because |e^{ikx}| = 1$

$= \int_{-\infty}^{\infty} |f(x)| dx$

$\Rightarrow \int_{-\infty}^{\infty} |e^{ikx} f(x)| dx = \int_{-\infty}^{\infty} |f(x)| dx$

Since integral on R.H.S. exists so integral of on L.H.S. also exists. So

$e^{ikx} f(x)$ is absolutely ~~convergent, so~~ ~~on $]-\infty, \infty[$~~ integrable on $]-\infty, \infty[$ and is therefore integrable over $]-\infty, \infty[$, i.e. $\int_{-\infty}^{\infty} e^{ikx} f(x) dx$ exists $\Rightarrow F\{f(x)\}$ exists.

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FOURIER TRANSFORMS OF SOME Simple Functions

Exp. 1:-

Evaluate F.T. of Gaussian Function defined by

$$g(x) = N e^{-\alpha x^2}$$

Where N and α are constants and $\alpha > 0$

Sol:-

$$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \cdot N e^{-\alpha x^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - \alpha x^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{ik}{\alpha} x\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - 2\left(\frac{ik}{2\alpha}\right)x + \left(\frac{ik}{2\alpha}\right)^2 - \left(\frac{ik}{2\alpha}\right)^2\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[\left(x - \frac{ik}{2\alpha}\right)^2 + \frac{k^2}{4\alpha^2}\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x - \frac{ik}{2\alpha}\right]^2} \cdot e^{-\frac{k^2}{4\alpha}} dx$$

$$= \frac{N}{\sqrt{2\pi}} \cdot e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right)\right]^2} dx$$

Put $\sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right) = y$

$\sqrt{\alpha}\, dx = dy \Rightarrow dx = \frac{1}{\sqrt{\alpha}} dy$

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So $F\{g(x)\} = \frac{N}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{\sqrt{\alpha}} dy$

$= \frac{N}{\sqrt{2\alpha\pi}} e^{-k^2/4\alpha} \sqrt{\pi}$ $\quad \because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

$\Rightarrow F\{N e^{-\alpha x^2}\} = \frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$

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Exp. 2:-

Calculate F.T. of

$g(x) = \frac{a}{x^2+a^2}$, $a > 0$

Sol.

$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik.x} g(x) dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \cdot \frac{a}{x^2+a^2} dx$

Clearly the Integral on the R.H.S. leads to the contour integration. So we use Cauchy's Residue Theorem to evaluate the integral.

Here arises three cases.

CASE: 1: If $k=0$

Then $F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{0} \cdot \frac{a}{x^2+a^2} dx$

$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx$

$= \frac{1}{\sqrt{2\pi}} \text{Tan}^{-1}\left(\frac{x}{a}\right) \Big|_{-\infty}^{\infty}$

$= \frac{1}{\sqrt{2\pi}} \left[\text{Tan}^{-1}(\infty) - \text{Tan}^{-1}(-\infty)\right]$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{\frac{\pi}{2}}$

CASE 2:- If $k>0$

If $k>0$, then singularities lie on the upper half plane.

The singularities are at where $x^2+a^2=0 \Rightarrow x^2=-a^2 \Rightarrow x=\pm ia$

Since $a>0$, so the only singularity which lie on upper half plane is ai.

Residue at $x=ai = \lim_{x\to ai} (x-ai) \cdot \dfrac{a e^{ikx}}{(x-ai)(x+ai)}$

$$= \frac{e^{-ka}}{2i}$$

So $F\{g(x)\} = \dfrac{1}{\sqrt{2\pi}} \cdot 2\pi i \cdot \sum R$

$$= \sqrt{2\pi} \cdot i \cdot \frac{e^{-ka}}{2i} = \sqrt{\frac{\pi}{2}} \cdot e^{-ka}$$

CASE:- 3:- If $k<0$

Then singularities lie in lower half plane. But the only singularity which lies on lower half plane is $x=-ai$

Residue at $x=-ai = \lim_{x\to -ai} \dfrac{(x+ai)\, a e^{ikx}}{(x-ai)(x+ai)}$

$$= \frac{e^{ka}}{-2i}$$

So $F\{g(x)\} = \dfrac{1}{\sqrt{2\pi}} \cdot -2\pi i \cdot \dfrac{e^{ka}}{-2i} = \sqrt{\dfrac{\pi}{2}}\, e^{k a}$

Combining all the above cases we have

$$F\{g(x)\} = \sqrt{\frac{\pi}{2}}\; e^{-|k|a} \quad \text{for all } k$$

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$$|x| = \begin{cases} 0 & \text{if } x=0 \\ x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$$

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Exp. 3:- Find F.T. of box function

$$f(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| \le a \end{cases}, \quad a > 0$$

Sol.

Given $f(x) = \begin{cases} 1 & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} e^{ikx} f(x)\,dx + \int_{-a}^{a} e^{ikx} f(x)\,dx + \int_{a}^{\infty} e^{ikx} f(x)\,dx \right]$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx} \cdot 1 \cdot dx + 0$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{ikx}}{ik} \Big|_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{1}{ik} \left[e^{ika} - e^{-ika} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{k} \left[\frac{e^{ika} - e^{-ika}}{2i} \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin ka}{k}$$

Exp. 4:- Show that $F\{x^n f(x)\} = (-i)^n \frac{d^n}{dk^n}(F(k))$

Sol.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$\frac{d}{dk}(F(k)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix) e^{ikx} f(x)\,dx = i\, F\{x f(x)\}$$

$$\Rightarrow \frac{1}{i} \frac{d}{dk}(F(k)) = F\{x f(x)\}$$

$$\Rightarrow (-i)^1 \cdot \frac{d}{dk}(F(k)) = F\{x f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} x f(x)\,dx$$

$$\Rightarrow (-i)^1 \frac{d^2}{dk^2}(F(k)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix) e^{ikx} x f(x)\,dx$$

$$\Rightarrow (-i)^2 \frac{d^2}{dk^2}(F(k)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (x^2 f(x))\,dx$$

$$\Rightarrow (-i)^2 \frac{d^2}{dk^2}(F(k)) = F\{x^2 f(x)\}$$

Continuing this process and differentiating upto n times we get

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{dk^n}(F(k))$$

Exp. 5:- Find F.T. of

$$f(x) = e^{-\lambda x^2}\cos\beta x \qquad (\lambda > 0)$$

Sol.

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{+ikx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{+ikx} e^{-\lambda x^2}\cos\beta x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - \frac{ik}{\lambda}x\right]}\cos\beta x\,dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - \frac{ik}{\lambda}x\right]} \cdot e^{i\beta x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - i\left(\frac{\beta+k}{\lambda}\right)x\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 + 2\left(\frac{\beta+k}{2i\lambda}\right)x\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\int_{-\infty}^{\infty} e^{-\lambda\left[x + \frac{\beta+k}{2i\lambda}\right]^2 + \lambda\left(\frac{\beta+k}{2i\lambda}\right)^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\, e^{-\lambda\left(\frac{\beta+k}{2\lambda}\right)^2}\int_{-\infty}^{\infty} e^{-\lambda\left[x + \frac{\beta+k}{2i\lambda}\right]^2}\,dx$$

Put $\sqrt{\lambda}\left[x + \frac{\beta+k}{2i\lambda}\right] = y$

$$dx = \frac{1}{\sqrt{\lambda}}\,dy$$

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So $F\{f(x)\} = \frac{e^{-\lambda\left(-\frac{\beta+k}{2\lambda}\right)^2}}{\sqrt{2\pi\lambda}} \, Re \int_{-\infty}^{\infty} e^{-y^2} dy$

$= \frac{e^{-\frac{(\beta+k)^2}{4\lambda}}}{\sqrt{2\pi\lambda}} \, Re \, (\sqrt{\pi})$

Hence $F\{f(x)\} = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(\beta+k)^2}{4\lambda}}$

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Boundedness and Continuity

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Theorem:-

If $f(x)$ is piece-wise smooth and absolutely integrable on the interval $]-\infty, \infty[$, then its F.T. $F(k)$ is bounded and continuous.

Proof:- Since $f(x)$ is absolutely integrable on $]-\infty, \infty[$ so

$$\int_{-\infty}^{\infty} |f(x)|\,dx \text{ exists.}$$

Let $\int_{-\infty}^{\infty} |f(x)|\,dx = A$

Now by definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$|F(k)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{ikx} \cdot f(x)\,dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{ikx}|\,|f(x)|\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\,dx$$

$$= \frac{1}{\sqrt{2\pi}} A$$

$$\Rightarrow |F(k)| \leq \frac{A}{\sqrt{2\pi}}$$

$\Rightarrow F(k)$ is bounded.

Now $\lim_{h\to 0} F(k+h) = \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k+h)x} f(x)\,dx$

$$= \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \cdot e^{ihx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left(\lim_{h\to 0} e^{ihx}\right) f(x)\,dx$$

[$\because$ h and x are independent]

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (1) f(x) dx$$

$$\Rightarrow \lim_{h \to 0} F(k+h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

$$\Rightarrow \lim_{h \to 0} F(k+h) = F(k)$$

$$\Rightarrow F(k) \text{ is continuous.}$$

Riemann-Lebesgue Theorem.

If $f(x)$ is piece-wise smooth and absolutely integrable function, then

$$\lim_{|k| \to \infty} F(k) = 0$$

Proof: By definition

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(x) \cdot \frac{e^{ikx}}{ik} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \frac{e^{ikx}}{ik} dx \right]$$

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \left[\lim_{x \to \infty} \frac{|f(x)|}{|ik|} |e^{ikx}| - \lim_{x \to -\infty} \frac{|f(x)|}{|ik|} |e^{ikx}| - \int_{-\infty}^{\infty} |f'(x)| \frac{|e^{ikx}|}{|ik|} dx \right]$$

Since $f(x)$ is absolutely integrable

So $\lim_{|x| \to \infty} |f(x)| = 0$

$$\Rightarrow |F(k)| \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{|k|} \int_{-\infty}^{\infty} |f'(x)| dx$$

$$\Rightarrow \lim_{|k| \to \infty} |F(k)| \leq 0$$

$$\Rightarrow \lim_{|k| \to \infty} F(k) = 0$$

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Fourier Transforms of Derivatives

Theorem.

If $f(x), f'(x), f''(x), \dots, f^{n-1}(x) \to 0$ as $x \to \pm\infty$, then

$$F\{f^{(n)}(x)\} = (-ik)^n F(k)$$

Proof:-

We prove the theorem by Principle of Mathematical Induction.

For $n=1$.

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{ikx} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (ik) e^{ikx} f(x)\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - 0 - \int_{-\infty}^{\infty} (ik) e^{ikx} f(x)\,dx \right]$$

$$= (-ik)^1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$= (-ik)^1 F(k) \qquad \Rightarrow F\{f'(x)\} = (-ik)^1 F(k)$$

So C-1 is satisfied.

Now suppose theorem is true for $n=m$ i.e.

$$F\{f^m(x)\} = (-ik)^m F(k) \to (1)$$

Now we prove the theorem for $n=k+1$.

$$F\{f^{m+1}(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f^{(m+1)}(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{ikx} f^m(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (ik) e^{ikx} f^m(x)\,dx \right]$$

$$= (-ik) F\{f^m(x)\} = (-ik)(-ik)^m F(k) \quad \text{by (1)}$$

$$\Rightarrow F\{f^{m+1}(x)\} = (-ik)^{m+1} F(k)$$

C-2 is satisfied. So theorem is true for all n.

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Convolution

The convolution of two functions $f(x)$ and $g(x)$, defined over $]-\infty,\infty[$, is denoted and defined by

$$f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, g(x-\eta)\, d\eta$$

Convolution Theorem:-

If $F(k)$ and $G(k)$ are Fourier Transforms of $f(x)$ and $g(x)$, then

$$F\{f*g\} = F(k)\,G(k)$$

or

$$F^{-1}\{F(k)\,G(k)\} = f*g$$

پروف :-

We prove $F^{-1}\{F(k)G(k)\} = f*g$

$$F^{-1}\{F(k)\,G(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k)\,G(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} g(x')\,dx' \right) dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x')} F(k)\,dk \right\} g(x')\,dx'$$

(Interchanging order of integration)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx' = g*f = f*g$$

Parseval's Theorem

(French Mathematician Chenes Parseval (1755 − 1836))

Parseval's 1st and 2nd theorems are given by

$$\int_{-\infty}^{\infty} |F(k)|^2 dk = \int_{-\infty}^{\infty} |f(x)| dx$$

and

$$\int_{-\infty}^{\infty} F(k) G(k) dk = \int_{-\infty}^{\infty} f(x) g(-x) dx$$

Proof:

First we prove Parseval's 2nd theorem, 1st theorem is a special case of 2nd theorem.

By definition of Convolution Theorem

$$F^{-1}\{F(k) G(k)\} = f(x) * g(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k) G(k)\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, g(x-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ikx} F(k) G(k)\, dk = \int_{-\infty}^{\infty} f(\eta)\, g(x-\eta)\, d\eta$$

Put $x = 0$

$$\Rightarrow \int_{-\infty}^{\infty} e^{0} F(k) G(k)\, dk = \int_{-\infty}^{\infty} f(\eta)\, g(0-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} F(k) G(k)\, dk = \int_{-\infty}^{\infty} f(\eta)\, g(-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} F(k) G(k)\, dk = \int_{-\infty}^{\infty} f(x)\, g(-x)\, dx$$

Which is req. Parseval's 2nd Theorem

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Take $g(-x) = \bar{f}(x)$

$\Rightarrow g(x) = \bar{f}(-x)$

$F\{g(x)\} = F\{\bar{f}(-x)\}$

$$\Rightarrow G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\bar{f}(-x)\,dx$$

Put $x = -x'$, $dx = -dx'$

$x \to \infty \Rightarrow x' = -\infty$

$x = -\infty \Rightarrow x' = \infty$

So

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ikx'}\,\bar{f}(x')\,(-dx')$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\,\bar{f}(x')\,dx'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\bar{f}(x)\,dx$$

$$= \bar{F}(k)$$

So

$$\int_{-\infty}^{\infty} F(k)G(k)\,dk = \int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} F(k)\bar{F}(k)\,dk = \int_{-\infty}^{\infty} f(x)\bar{f}(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(k)|^2\,dk = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$\because z\bar{z} = |z|^2$

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If $f(x)$ is a real function defined over $]-\infty,\infty[$ and the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(x')\cos k(x-x')\,dx'\,dk$$

Proof:- Since $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent so F.T. of $f(x)$ and corresponding inverse F.T. exists.

So by definition of I.F.T.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-ikx} F(k)dk + \int_0^{\infty} e^{-ikx} F(k)\,dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{ik'x} F(-k')(-dk') + \int_0^{\infty} e^{-ikx} F(k)dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ik'x} F(-k')\,dk' + \int_0^{\infty} e^{-ikx} F(k)dk\right]$$

Since $f(x)$ is real, So by conjugation property $F(-k') = \bar{F}(k') = \bar{F}(k)$

So

$$f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ik'x}\bar{F}(k')dk' + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ikx}\bar{F}(k)dk + \int_0^{\infty} e^{-ikx}F(k)\,dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[e^{ikx}\bar{F}(k) + e^{-ikx}F(k)\right]dk \longrightarrow (1)$$

Now $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')\,dx'$

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$$\Rightarrow e^{-ikx} F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \cdot e^{ikx'} f(x') dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x')} f(x') dx'$$

$$\Rightarrow e^{ikx} \bar{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x')} f(x') dx'$$

$\because f$ is real.

So by ①

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x')} f(x') dx' + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x')} f(x') dx' \right] dk$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \left[e^{ik(x-x')} + e^{-ik(x-x')} \right] f(x') dx' dk$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \left[\frac{e^{ik(x-x')} + e^{-ik(x-x')}}{2} \right] f(x') dx' dk$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \cos k(x-x') f(x') dx' dk$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(x') \cos k(x-x') dx' dk$$

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Fourier Sine and Cosine Transforms.

If a function $f(x)$ is defined over the interval $[0, \infty[$, then we can define Forier Sine and Fourier Cosine Transforms as

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx$$

and $$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx$$

respectively.

The corresponding Inverse Fourier Sine and Inverse Fourier Cosine Transforms are

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \, dk$$

and $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx \, dk$$

JUSTIFICATION FOR THE DEFINITIONS.

The above definitions are directly follows from the def. of complex or exponentional F.T. Let a real valued function $f(x)$ be defined over $[0, \infty[$.

Now define a function

$$f_c(x) = \begin{cases} f(x) & \text{for } 0 \le x < \infty \\ f(-x) & \text{for } -\infty < x \le 0 \end{cases}$$

Then $f_c(x)$ is the extension of $f(x)$.

Now $$F\{f_c(x)\} = F_c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f_c(x) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{ikx} f_c(x) \, dx + \int_0^\infty e^{ikx} f_c(x) \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{ikx} f(-x) \, dx + \int_0^\infty e^{ikx} f(x) \, dx \right]$$

put $x = -x'$ in 1st integral

$$F_c(k) = \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{-ikx'} f(x')\, d(-x') + \int_0^{\infty} e^{ikx} f(x)\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-ikx} f(x)\, dx + \int_0^{\infty} e^{ikx} f(x)\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^{\infty}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right) f(x)\, dx\right]$$

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx\, dx$$

Fourier Sine and Cosine Transforms of Derivatives.

One:- If $f(x)$ is real valued and $|f(x)| \to 0$ as $x \to \infty$, Then

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\cos kx\, f(x)\Big|_0^{\infty} - \int_0^{\infty} f(x)(-k\sin kx)\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-f(0) + k\int_0^{\infty}\sin kx\, f(x)\, dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + k F_s(k)$$

Two:- $F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\sin kx\, dx$

$$= \sqrt{\frac{2}{\pi}}\left[\sin kx\, f(x)\Big|_0^{\infty} - \int_0^{\infty} k\cos kx\, f(x)\, dx\right]$$

$$= -k\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos kx\, f(x)\, dx = -k F_c(k)$$

$$\Rightarrow F_s\{f'(x)\} = -k F_c(k)$$

Three: If $f(x), f'(x) \to 0$ as $x \to \infty$, then

$$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f''(x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin kx \, f'(x) \Big|_0^\infty - \int_0^\infty k \cos kx \, f'(x)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - k \left\{ \cos kx \, f(x) \Big|_0^\infty - \int_0^\infty (-k \sin kx) f(x)\, dx \right\} \right]$$

$$= k\sqrt{\frac{2}{\pi}} f(0) - k^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f(x)\, dx$$

$$= k\sqrt{\frac{2}{\pi}} f(0) - k^2 F_s(k)$$

چار:- $F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \cos kx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\cos kx \, f'(x) \Big|_0^\infty - \int_0^\infty (-k \sin kx) f'(x)\, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + \sqrt{\frac{2}{\pi}} k \left[\sin kx \, f(x) \Big|_0^\infty - \int_0^\infty k \cos kx \, f(x)\, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) - \sqrt{\frac{2}{\pi}} k^2 \int_0^\infty \cos kx \, f(x)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) - k^2 F_c(k)$$

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Illustrative Examples

Example 1:

Calculate $F_s(k)$ s.t. $f(x) = e^{-x}\cos x$

Sol.

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin kx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}\cos x \sin kx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\infty e^{-x}(2\sin kx \cos x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x}(\sin(k+1)x + \sin(k-1)x)\,dx$$

Using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$

And $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$ $\quad a = -1,\ b = k+1$.

$$F_s(k) = \frac{1}{\sqrt{2\pi}}\left[e^{-x}\left\{\frac{1}{(k+1)^2+1}\left(-\sin(k+1)x - (k+1)\cos(k+1)x\right) + \frac{1}{(k-1)^2+1}\left(-\sin(k-1)x - (k-1)\cos(k-1)x\right)\right\}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \left\{\frac{1}{(k+1)^2+1}(0-(k+1)) + \frac{1}{(k-1)^2+1}(0-(k-1))\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{k+1}{(k+1)^2+1} + \frac{k-1}{(k-1)^2+1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{k+1}{k^2+2k+2} + \frac{k-1}{k^2-2k+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2k^3}{k^4+4}$$

$$\Rightarrow F_s\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}} \cdot \frac{k^3}{k^4+4}$$

Example 2:-

If $f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$

Then Calculate $F_s(k)$

Sol.

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin kx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{-2} \int_0^\pi (-2 \sin kx \sin x) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\pi (\cos(k+1)x - \cos(k-1)x) \, dx$$

$$= -\frac{1}{\sqrt{2\pi}} \left. \frac{\sin(k+1)x}{k+1} - \frac{\sin(k-1)x}{k-1} \right|_0^\pi$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{\sin(k+1)\pi}{k+1} - \frac{\sin(k-1)\pi}{k-1} - 0 + 0 \right]$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{(k-1)[\sin(k+1)\pi] - (k+1)\sin(k-1)\pi}{k^2-1} \right]$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{(k-1)\sin k\pi - (k+1)\sin k\pi}{k^2-1} \right]$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{-k\sin k\pi + \sin k\pi - k\sin k\pi - \sin k\pi}{k^2-1} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2k\sin k\pi}{k^2-1}$$

$$= \frac{2k \sin k\pi}{\sqrt{2\pi}\,(k^2-1)}$$

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EXAMPLE :-

Calculate Fourier Sine Transform of the following

$$f(x) = x e^{-ax}$$

Solution.

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \cdot x e^{-ax}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \left(e^{-ax} \sin kx\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \left\{ \frac{e^{-ax}}{a^2+k^2} (-a \sin kx - k \cos kx) \right\} \Big|_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-ax}}{a^2+k^2} (-a \sin kx - k \cos kx)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \frac{1}{a^2+k^2} \left\{ a \int_0^\infty e^{-ax} \sin kx\, dx + k \int_0^\infty e^{-ax} \cos kx\, dx \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+k^2} \left\{ \frac{e^{-ax}}{a^2+k^2} (-a \sin kx - k \cos kx) \right\} + \frac{k}{a^2+k^2} \left\{ \frac{e^{-ax}}{a^2+k^2} (-a \cos kx + k \sin kx) \right\} \right] \Big|_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{a}{a^2+k^2} \left\{ \frac{1}{a^2+k^2} (0 - k) \right\} - \frac{k}{a^2+k^2} \left\{ \frac{1}{a^2+k^2} (-a + 0) \right\} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{+ak}{(a^2+k^2)^2} + \frac{ak}{(a^2+k^2)^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2ak}{(a^2+k^2)^2}$$

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Example 4:-

Calculate $F_c(k)$ $f(x) = x^{\alpha-1}$ $\quad 0<\alpha<1$

Proof:-

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \cos kx\,dx \rightarrow (1)$$

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \sin kx\,dx \rightarrow (2)$$

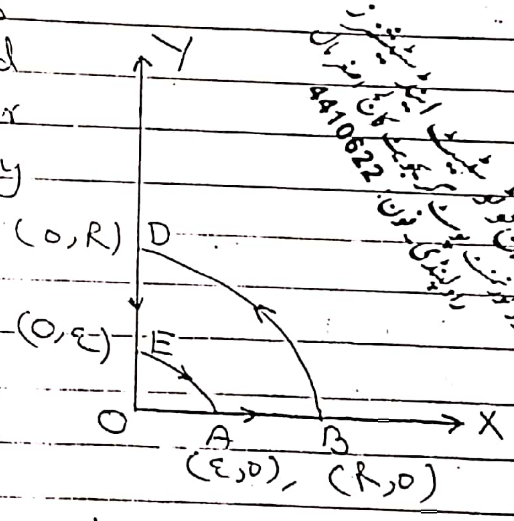
Define a function

$$f(z) = z^{\alpha-1} e^{-kz}, \quad 0<\alpha<1$$

and choose the contour as shown in figure.

Then $f(z)$ is analytic on and within the contour C, so then by Cauchy's theorem

$$\int_C f(z)\,dz = 0$$



$$\Rightarrow \int_{C_1} f(z)\,dz + \int_\varepsilon^R f(z)\,dz + \int_{C_2} f(z)\,dz + \int_R^\varepsilon f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_\varepsilon^R x^{\alpha-1} e^{-kx}\,dx + \int_{C_2} f(z)\,dz + \int_R^\varepsilon (iy)^{\alpha-1} . e^{-iky} (i\,dy) = 0$$

When $\varepsilon \rightarrow 0$, $R \rightarrow \infty$, then

$$\int_{C_1} f(z)\,dz = 0 \quad \& \quad \int_{C_2} f(z)\,dz = 0$$

Then $\displaystyle\int_0^\infty x^{\alpha-1} e^{-kx}\,dx = i\int_0^\infty (iy)^{\alpha-1} e^{-iky}\,dy$

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$$\Rightarrow \int_0^\infty x^{\alpha-1} e^{-kx}\, dx = \int_0^\infty i^\alpha y^{\alpha-1} (e^{-iky})\, dy$$

Now $(i)^\alpha = \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^\alpha = e^{i\alpha\pi/2}$

$$\Rightarrow \int_0^\infty x^{\alpha-1} e^{-kx}\, dx = \int_0^\infty e^{i\alpha\pi/2}\, y^{\alpha-1} e^{-iky}\, dy$$

$$\Rightarrow e^{-\frac{i\alpha\pi}{2}} \int_0^\infty x^{\alpha-1} e^{-kx}\, dx = \int_0^\infty y^{\alpha-1} e^{-iky}\, dy$$

Equating real and imaginary parts

$$\cos\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\, dx = \int_0^\infty y^{\alpha-1} \cos ky\, dy \rightarrow (3)$$

and $\sin\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\, dx = \int_0^\infty y^{\alpha-1} \sin ky\, dy \rightarrow (4)$

From (1) and (3)

$$F_c(k) = \sqrt{\frac{2}{\pi}} \cos\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\, dx$$

Put $kx = t$

$$F_c(k) = \sqrt{\frac{2}{\pi}} \cos\frac{\alpha\pi}{2} \int_0^\infty (t/k)^{\alpha-1} e^{-t} \frac{1}{k}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\cos\frac{\alpha\pi}{2}}{k^\alpha} \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\cos\frac{\alpha\pi}{2}}{k^\alpha}\, \Gamma(\alpha)$$

Now from (2) and (4)

$$F_s(k) = \sqrt{\frac{2}{\pi}} \sin\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\frac{\alpha\pi}{2}}{k^\alpha}\, \Gamma(\alpha)$$

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HOUSE OF QUALITY NOTES

Exp. 5 :-

Show that

$$F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-k^2}{(a^2+k^2)^2}$$

Sol.

$$F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-ax} \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \cdot \frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\Big|_0^\infty - \int_0^\infty \frac{e^{-ax}}{a^2+k^2}(\cdots\cdots) dx \right.$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a}{a^2+k^2} \int_0^\infty e^{-ax} \cos kx \, dx + \frac{k}{a^2+k^2} \int_0^\infty e^{-ax} \sin kx \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a}{a^2+k^2} \left\{ \frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx) \right\} \Big|_0^\infty \right.$$

$$\left. + \frac{k}{a^2+k^2} \left\{ \frac{e^{-ax}}{a^2+k^2}(-a\sin kx - k\cos kx) \right\} \Big|_0^\infty \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a^2-k^2}{(a^2+k^2)^2}$$

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Use of $F_s(K)$ and $F_c(K)$ in B.V. and I.V. PROBLEMS

Exp. 1:-

Solve the potentional eq. in the semi-infinite strip $0<x<c$, $y>0$ so that satisfies the following conditions.

$u(0,y)=0$, $u_y(x,0)=0$, $u_x(c,y)=f(y)$

Sol.

The potential eq. is given by

$$u_{xx}+u_{yy}=0$$

Now Applying F.C.T. w.r.t. y.

$$F_c\{u_{xx}\}+F_c\{u_{yy}\}=0$$

$$\frac{d^2}{dx^2}U_c(x,k)-\sqrt{\frac{2}{\pi}}\,u_y(x,0)-k^2U_c(x,k)=0$$

$$\Rightarrow\left(\frac{d^2}{dx^2}-k^2\right)U_c(x,k)-0=0$$

A.E. is $\frac{d^2}{dx^2}-k^2=0$

$D^2=\frac{d^2}{dx^2}=k^2$

$D=\pm k$

$$\Rightarrow U_c(x,k)=c_1e^{kx}+c_2e^{-kx}\rightarrow(1)$$

Now $u(0,y)=0$

$\Rightarrow U_c(0,k)=0$

So $c_1+c_2=0\rightarrow(2)$ by (1)

Also $u_x(c,y)=f(y)$

$\Rightarrow \frac{d}{dx}U_c(c,k)=F_c(k)\rightarrow(3)$

Now by (1)

$$\frac{d}{dx}U_c(x,k)=c_1ke^{kx}-c_2ke^{-kx}$$

$$\frac{d}{dx}U_c(c,k)=c_1ke^{ck}-c_2ke^{-ck}$$

$$\Rightarrow c_1ke^{ck}-c_2ke^{-ck}=F_c(k)\rightarrow(4)\text{ by }(3)$$

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Solving ② and ④

$C_1 = \frac{2F_c(k)}{k\cosh ck}$, $C_2 = -\frac{2F_c(k)}{k\cosh ck}$

Hence $U_c(x,k) = \frac{2F_c(k)e^{kx}}{k\cosh ck} - \frac{2F_c(k)e^{-kx}}{k\cosh ck}$

$= \frac{2F_c(k)}{k\cosh ck}\left[\frac{e^{kx}-e^{-kx}}{2}\right] \times 2$

$= \frac{4F_c(k)\sinh kx}{k\cosh ck}$

Taking I.F.C.T.

$u(x,y) = \sqrt{\frac{2}{\pi}} \cdot 4\int_0^\infty \frac{F_c(k)\sinh kx}{k\cosh ck} \cos ky\, dk$

$= \sqrt{\frac{2}{\pi}}\, 4\int_0^\infty \frac{\sinh kx \cos ky}{k\cosh ck}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty \cos ky' f(y')\, dy'\right) dk$

$\Rightarrow u(x,y) = \frac{8}{\pi}\int_0^\infty\int_0^\infty \frac{\sinh kx \cos ky \cos ky'}{k\cosh ck} f(y')\, dy'\, dk$

If $f(y')$ is known then on further integrating $u(x,y)$ can be obtained.

Example 2.

Solve

$u_t = u_{xx}$ $x \geq 0, t \geq 0$

$u(0,t) = u_0$, $u(x,0) = 0$, $u_0 \geq 0$

Sol.

B.C. Suggests that we should apply ~~$F_s(k)$~~ F.S.T. w.r.t. x

$F_s\{u_t\} = F_s\{u_{xx}\}$

$\frac{d}{dt}\{U_s(k,t)\} = \sqrt{\frac{2}{\pi}}\, k\, u(0,t) - k^2 U_s(k,t)$

$= \sqrt{\frac{2}{\pi}}\, k\, u_0 - k^2 U_s(k,t)$

HOUSE OF QUALITY NOTES

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$$\Rightarrow \frac{d}{dt} U_s(k,t) + k^2 U_s(k,t) = \sqrt{\frac{2}{\pi}}\, k u_0$$

$$I.F. = e^{\int k^2 dt} = e^{k^2 t}$$

So $d\left(U_s(k,t)\, e^{k^2 t}\right) = \sqrt{\frac{2}{\pi}}\, k u_0 e^{k^2 t} dt$

Integrating both sides

$$U_s(k,t)\, e^{k^2 t} = \sqrt{\frac{2}{\pi}}\,(1/k)\, u_0 e^{k^2 t} + C$$

Now $u(x,0) = 0 \Rightarrow U_s(k,0) = 0$

So $U_s(k,0) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} + C$

$$\Rightarrow C = -\sqrt{\frac{2}{\pi}}\, \frac{u_0}{k}$$

$$\Rightarrow e^{k^2 t} U_s(k,t) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} e^{k^2 t} - \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k}$$

$$\Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{k} - \sqrt{\frac{2}{\pi}}\, u_0 \frac{1}{k} e^{-k^2 t}$$

Taking Inverse Fourier Sine Transform on both sides

~~$u(x,t) = \sqrt{\frac{2}{\pi}}\, u_0 \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{k^2 t}}{k} dk - \sqrt{\frac{1}{\pi}} \cdot \sqrt{\frac{2}{\pi}}$~~

$$u(x,t) = \sqrt{\frac{2}{\pi}}\, u_0 \left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin kx}{k} dk - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-k^2 t}}{k} \sin kx\, dk\right]$$

$$= \frac{2}{\pi} u_0 \left[\int_0^\infty \frac{\sin kx}{k} dk - \int_0^\infty \frac{1}{k} e^{-k^2 t} \sin kx\, dk\right]$$

$$= \sqrt{\frac{2}{\pi}}\, u_0 \left[I_1 - I_2\right] \rightarrow (a)$$

$$I_1 = \int_0^\infty \frac{\sin kx}{k} dk = \frac{\pi}{2}$$

$$I_2 = \int_0^\infty e^{-k^2 t} \frac{\sin kx}{k} dk$$

$$= \int_0^\infty e^{-k^2 t} \left(\int_0^x \cos kx'\, dx'\right) dk$$

$$\because \frac{\sin kx}{k} = \int_0^x \cos kx'\, dx'$$

$$I_2 = \int_0^x \left[\int_0^\infty e^{-k^2 t} \cos kx'\, dx'\right] dk \rightarrow (1)$$

Consider $\int_0^\infty e^{-k^2 t} \cos kx'\, dx'$

$$= \mathrm{Re} \int_0^\infty e^{-k^2 t} e^{ikx'} dx'$$

$$= Re\int_0^\infty e^{-k^2 t} e^{ikx'} dk$$

$$= Re\int_0^\infty e^{-t[k^2 - \frac{ikx'}{t}]} dk$$

$$= Re\int_0^\infty e^{-t(k-\frac{ix'}{2t})^2} \cdot e^{-x'^2/4t} dk$$

$$= e^{-\frac{x'^2}{4t}} Re\int_0^\infty e^{-t(k-ix'/2t)^2} dk$$

$$-t[k^2 - \frac{ikx'}{t}]$$

$$= -t\left[k^2 - 2\left(\frac{ix'}{2t}\right)(k) + \left(\frac{ix'}{2t}\right)^2 - \left(\frac{ix'}{2t}\right)^2\right]$$

$$= -\left(k - \frac{ix'}{2t}\right)^2 - \frac{x'^2}{4t}$$

Put $k - ix'/2t = k' \Rightarrow dk = dk'$

$$= e^{-x'^2/4t} Re\int_0^\infty e^{-tk'^2} dk = e^{-x'^2/4t} \cdot \sqrt{\pi/t}$$

Thus by ① $I_2 = \int_0^x e^{-x'^2/4t} \sqrt{\pi/t}\, dx'$

$$= \frac{\sqrt{\pi}}{t} \int_0^x e^{-x'^2/4t} dx'$$

$$= \frac{\sqrt{\pi}}{t} \, erf\left(\frac{x}{2\sqrt{t}}\right)$$

$$\left(\because erf(x) = \int_0^x e^{-t^2} dt\right)$$

So by (a)

$$u(x,t) = \frac{2}{\pi} u_0 \left(\frac{\pi}{2} - \frac{\sqrt{\pi}}{t} erf\left(\frac{x}{2\sqrt{t}}\right)\right)$$

$$= u_0\left(1 - \frac{2}{\sqrt{\pi}\, t} erf\left(\frac{x}{2\sqrt{t}}\right)\right)$$

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USE OF COMPLEX FOURIER TRANSFORM IN SOLVING B.V., I.V. PROBLEMS

Exp. 1. Solve the problem by using F.T. Method

$$u_{xx}(x,t) = u_t(x,t)$$
$$u(x,0) = e^{-\alpha x^2}$$

where $u(x,t), u_x(x,t) \to 0$ as $x \to \pm\infty$

Sol. $F\{u_{xx}(x,t)\} = F\{u_t(x,t)\}$

$$(-ik)^2 U(k,t) = \frac{d}{dt} U(k,t)$$
$$-k^2 U(k,t) = \frac{d}{dt} U(k,t)$$
$$\Rightarrow \frac{U'(k,t)}{U(k,t)} = -k^2$$
$$\Rightarrow \ln U(k,t) = -k^2 t + \ln A$$
$$= \ln e^{-k^2 t} + \ln A$$
$$\Rightarrow \ln U(k,t) = \ln A e^{-k^2 t}$$
$$\Rightarrow U(k,t) = A e^{-k^2 t} \quad \to (1)$$

Now $u(x,0) = e^{-\alpha x^2}$ Use Guassian function

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$$
$$\Rightarrow A = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha} \quad \text{by (1)}$$
$$\Rightarrow U(k,t) = \frac{1}{\sqrt{2\alpha}} e^{-k^2 t} \cdot e^{-k^2/4\alpha}$$
$$= \frac{1}{\sqrt{2\alpha}} e^{-k^2\left(\frac{1}{4\alpha} + t\right)}$$
$$= \frac{1}{\sqrt{2\alpha}} e^{-\beta k^2} \qquad \beta = \frac{1}{4\alpha} + t$$
$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\alpha}} F^{-1}\{e^{-\beta k^2}\}$$

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$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\alpha}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\beta k^2} dk$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\beta\left(k^2 + \frac{ikx}{\beta}\right)} dk$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\beta\left(k + \frac{ix}{2\beta}\right)^2} \cdot e^{\frac{-x^2}{4\beta}} dk$$

$$= \frac{1}{2\sqrt{\pi\alpha}} e^{-x^2/4\beta} \cdot \sqrt{\frac{\pi}{\beta}}$$

$$= \frac{1}{2\sqrt{\alpha\beta}} e^{-x^2/4\beta}$$

$$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\alpha\left(\frac{1}{4\alpha}+t\right)}} \cdot e^{-x^2/4\left(\frac{1}{4\alpha}+t\right)}$$

$$= \frac{1}{\sqrt{1+4\alpha t}} e^{-\alpha x^2/(1+4\alpha t)}$$

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Exp. 2:-

Solve by using F.T.M.

$u_t = u_{xx}$ $\quad x>0,\ t>0$

$u_x(0,t) = f(t)$ $\quad ,\ t>0$

$u(x,0) = 0$ $\quad ,\ x>0$

Sol.

The and B.C. Suggests that we should Apply Fourier Cosine Transforms

So $F_c\{u_t\} = F_c\{u_{xx}\}$

$$\Rightarrow \frac{d}{dt} U_c(k,t) = -\sqrt{\frac{2}{\pi}}\, u_x(0,t) - k^2 U_c(k,t)$$

$$= -\sqrt{\frac{2}{\pi}}\, f(t) - k^2 U_c(k,t)$$

$$\Rightarrow \frac{d}{dt} U_c(k,t) + k^2 U_c(k,t) = -\sqrt{\frac{2}{\pi}}\, f(t)$$

$$I.F. = e^{k^2 t}$$

$$\Rightarrow e^{k^2 t} U_c(k,t) = -\sqrt{\frac{2}{\pi}} \int e^{k^2 t} f(t)\, dt + C \quad \rightarrow (1)$$

Now $u(x,0) = 0$

$\Rightarrow U_c(k,0) = 0$

By (1) $\quad C = 0 \Rightarrow e^{k^2 t} U_c(k,t) = -\sqrt{\frac{2}{\pi}} \int e^{k^2 t} f(t)\, dt$

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$$\Rightarrow U_t(k,t) = -\sqrt{\frac{2}{\pi}}\, e^{-k^2 t}\int f(t)\, e^{k^2 t}\, dt$$

$$= -\sqrt{\frac{2}{\pi}}\, e^{-k^2 t}\int f(t')\, e^{k^2 t'}\, dt'$$

$$= -\sqrt{\frac{2}{\pi}}\int f(t')\, e^{k^2(t'-t)}\, dt'$$

$$\Rightarrow u(x,t) = -\sqrt{\frac{2}{\pi}}\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\int f(t')\, e^{k^2(t'-t)}\, dt'\right)\cos kx\, dk$$

$$\Rightarrow u(x,t) = -\frac{2}{\pi}\int f(t')\int_0^\infty e^{k^2(t'-t)}\cos kx\, dk\, dt'$$

$$= -\sqrt{\frac{2}{\pi}}\int f(t')\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{k^2(t'-t)}\cos kx\, dk\right] dt'$$

$$u(x,t) = -\sqrt{\frac{2}{\pi}}\int f(t')\, \frac{1}{t'-t}\, e^{\left(-x^2/4(t'-t)\right)}\, dt'$$

AS. REQ.

Exp. 3. Solve

$$u_t = \alpha^2 u_{xx} \qquad -\infty < x < \infty,\ t > 0$$

with $\quad u(x,0) = f(x) \qquad -\infty < x < \infty$

$\quad |u(x,0)| < \infty$ for all x, t

Sol.

Given $u_t = \alpha^2 u_{xx}$

$$F\{u_t\} = \alpha^2 F\{u_{xx}\}$$

$$\Rightarrow U'(k,t) = \alpha^2(-ik)^2\, U(k,t)$$

$$\Rightarrow \frac{U'(k,t)}{U(k,t)} = -\alpha^2 k^2$$

$$\Rightarrow \ln U(k,t) = -\alpha^2 k^2 t + \ln A$$

$$= \ln A e^{-\alpha^2 k^2 t}$$

$$\Rightarrow U(k,t) = A e^{-\alpha^2 k^2 t}$$

$$U(k,0) = A$$

Also $u(x,0) = f(x)$

$$\Rightarrow U(k,0) = F(k) \Rightarrow A = F(k)$$

$$\Rightarrow U(k,t) = F(k)\, e^{-\alpha^2 k^2 t}$$

Applying F^{-1}

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} F(k)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} f(x')\, dx' \right) dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} e^{ikx'}\, dk \right) dx'$$

Consider

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 t \left[k^2 + \frac{ikx}{\alpha^2 t} - \frac{ikx'}{\alpha^2 t} \right]} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 t \left[k^2 + 2\left(\frac{ix - ix'}{2\alpha^2 t} \right) k + \left\{ \frac{i(x-x')}{2\alpha^2 t} \right\}^2 - \left\{ \frac{i(x-x')}{2\alpha^2 t} \right\}^2 \right]} dk$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} \int_{-\infty}^{\infty} e^{-\alpha^2 t \left(k + \frac{i(x-x')}{2\alpha^2 t} \right)^2} dk$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} \int_{-\infty}^{\infty} e^{-\alpha^2 t k'^2} dk'$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(x-x')^2/4\alpha^2 t} \sqrt{\frac{\pi}{\alpha^2 t}}$$

$$= \frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{(x-x')^2}{4\alpha^2 t}}$$

So

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} dx'$$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x') e^{-\frac{(x-x')^2}{4\alpha^2 t}} dx'$$

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Exp. 4: Solve

$$u_t = u_{xx} \qquad 0<x<\infty,\ t>0$$
$$u_x(0,t) = 0 \qquad t>0$$
$$u(x,0) = f(x) \qquad 0<x<\infty$$

Sol.

As $0<x<\infty$, so we have to use Fourier sine or Fourier Cosine Transform. But 2nd B.C. suggests that we should use F.C.T.

$$F_c\{u_t\} = F_c\{u_{xx}\}$$
$$U_c'(k,t) = -\sqrt{\tfrac{2}{\pi}}\,u_x(0,t) - k^2 U_c(k,t)$$
$$= 0 - k^2 U_c(k,t)$$
$$\Rightarrow \ln U_c(k,t) = \ln A e^{-k^2 t}$$
$$\Rightarrow U_c(k,t) = A e^{-k^2 t}$$
$$U_c(k,0) = A$$

Now $u(x,0) = f(x)$

$$\Rightarrow U_c(k,0) = F_c(k)$$
$$\Rightarrow A = F_c(k)$$
$$\Rightarrow U_c(k,t) = F_c(k)\, e^{-k^2 t}$$

$$\Rightarrow u(x,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \cos kx\, F_c(k)\, e^{-k^2 t}\, dk$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty \cos kx \left[\sqrt{\tfrac{2}{\pi}} \int_0^\infty \cos kx'\, f(x')\, dx'\right] e^{-k^2 t}\, dk$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos kx \cos kx'\, f(x')\, e^{-k^2 t}\, dx'\, dk$$

A.S.R.E.Q.

4410622

20

Exp. 5:-

Solve

$$u_{xxxx} = \frac{1}{\alpha^2} u_{tt}$$

with $u(x,0) = f(x)$

$u_t(x,0) = a\, g'(x)$

and $g, u, u_{xx}, u_{xxx} \to 0$ as $x \to \pm\infty$

Sol. As $x \to \pm\infty$

So we have to apply F.T.

$$F\{u_{xxxx}\} = \frac{1}{\alpha^2} F\{u_{tt}\}$$

$$\Rightarrow (-ik)^4 U(k,t) = \frac{1}{\alpha^2} \frac{d^2}{dt^2} U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2} U(k,t) - \alpha^2 k^4 U(k,t) = 0$$

A.E. is $D^2 - \alpha^2 k^4 = 0 \Rightarrow D = \pm \alpha k^2$

$$\Rightarrow U(k,t) = A e^{\alpha k^2 t} + B e^{-\alpha k^2 t} \to (1)$$

$$U(k,0) = A + B \to (2)$$

$$U'(k,t) = \alpha k^2 (A e^{\alpha k^2 t} - B e^{-\alpha k^2 t})$$

$$U'(k,0) = \alpha k^2 (A - B) \to (3)$$

Now $u(x,0) = f(x) \Rightarrow U(k,0) = F(k)$

$u_t(x,0) = a\, g'(x)$

$$\Rightarrow \frac{d}{dt} U(k,0) = a(-ik)\, G(k)$$

By (2)

$$A + B = F(k)$$

By (3) $\alpha k^2 (A - B) = -iak\, G(k) \Rightarrow A - B = -\frac{i\alpha}{\alpha k} G(k)$

$$\Rightarrow A = \frac{1}{2}\left\{F(k) - \frac{i}{k} G(k)\right\} \to (4)$$

$$B = \frac{1}{2}\left\{F(k) + \frac{i}{k} G(k)\right\} \to (5)$$

Hence

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} (A e^{ak^2 t} + B e^{-ak^2 t})\, dk$$

Where A + B are given in (4) + (5)

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Exp. 6: Solve

$$u_{xx}(x,t) = \frac{1}{c^2} u_{tt}(x,t)$$

$u(x,0) = p(x)$ $\quad u_t(x,0) = v(x)$

$u, u_x \to 0$ as $x \to \pm\infty$

Sol.

As $x \to \pm\infty$, so we use Fourier Complex Transform

$$F\{u_{xx}\} = F\{u_{tt}\}$$

$$-k^2 U(k,t) = \frac{1}{c^2}\frac{d^2}{dt^2} U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2} U(k,t) + c^2k^2 U(k,t) = 0$$

A.E. is $D^2 + c^2k^2 = 0$

$D = \pm ick$

$$\Rightarrow U(k,t) = A\cos ckt + B\sin ckt$$

$U(k,0) = A$

$U_t(k,t) = -ckA\sin ckt + ckB\cos ckt$

$U_t(k,0) = ckB$

Now $u(x,0) = p(x) \Rightarrow U(k,0) = P(k)$

$u_t(x,0) = v(x) \Rightarrow U_t(k,0) = Q(k)$

So $A = P(k)$

$B = \frac{1}{ck} Q(k)$

So $U(k,t) = P(k)\cos ckt + \frac{1}{ck} Q(k)\sin ckt$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\left[P(k)\cos ckt + \frac{1}{ck} Q(k) \sin ckt\right] dk$$

Where

$$P(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} p(x)\,dx$$

& $$Q(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} v(x)\,dx$$

Multidimensional Fourier Transform

Double F.T. And Its Inverse:-

Let $f(x,y)$ be a function defined over the whole XY-plane i.e. $-\infty < x,y < \infty$ then its F.T. is denoted and defined as

$$F(k_1,k_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{ik_1x+ik_2y}\, dx\, dy$$

and I.F.T. is given by

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1,k_2)\, e^{-ik_1x-ik_2y}\, dk_1\, dk_2$$

Tripple F.T. & Its Inverse

With usual meanings

$$F(k_1,k_2,k_3) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} f(x,y,z)\, dx\, dy\, dz$$

and

$$f(x,y,z) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y+k_3z)} F(k_1,k_2,k_3)\, dk_1\, dk_2\, dk_3$$

Remarks:- In a similar way n-dimensional F.T. and correspondingly [illegible] I.F.T. [illegible]

Example:- Find F.T. of $\nabla^2 U = U_{xx} + U_{yy} + U_{zz}$

Sol.- It is trivial to show that

$$F\{\nabla^2 U\} = -(k_1^2+k_2^2+k_3^2)\, U(k_1,k_2,k_3)$$

Example 2.- Use F.T. method to solve the Poisson Equation $\nabla^2 U = g(x,y,z)$.

Sol.- $F\{\nabla^2 U\} = F\{g(x,y,z)\}$

$$\Rightarrow -(k_1^2+k_2^2+k_3^2)\, U(k_1,k_2,k_3) = G(k_1,k_2,k_3)$$

$$\Rightarrow U(K) = -\frac{G(K)}{K^2}$$

Applying I.F.T.

$$U(r) = -\frac{1}{(\sqrt{2\pi})^3}\int e^{-ik\cdot r}\, \frac{G(k)}{k^2}\, d^3k$$

$$\Rightarrow U(r) = -\frac{1}{(\sqrt{2\pi})^3}\int_V \frac{e^{-ik\cdot r}}{k^2}\left(\frac{1}{(\sqrt{2\pi})^3}\int_V e^{ik\cdot r'} g(r')\, d^3r'\right) d^3k$$

$$\Rightarrow U(r) = -\frac{1}{(\sqrt{2\pi})^6}\int_V\int_V e^{-ik\cdot(r-r')}\, g(r')\, d^3r'\, d^3k$$

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By changing order of integration

$$u(r) = -\frac{1}{(\sqrt{2\pi})^6}\int_V\int_V \frac{e^{-ik\cdot(r-r')}}{k^2}\, \rho(r')\, d^3k\, d^3r'$$

$$= -\frac{1}{(\sqrt{2\pi})^6}\int_V\left[\int_V \frac{e^{-ik(r-r')}}{k^2}\, d^3k\right]\rho(r')\, d^3r' \longrightarrow (*)$$

Now consider $I = \int_V \frac{e^{-ik\cdot(r-r')}}{k^2}\, d^3k$

$$\Rightarrow I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i[k_1(x-x')+k_2(y-y')+k_3(z-z')]}}{k_1^2+k_2^2+k_3^2}\, dk_1\, dk_2\, dk_3$$

By using spherical coordinates

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-ik\cdot R}}{k^2}\left(k^2 \sin\theta\, dk\, d\theta\, d\phi\right)$$ Here $R = r - r'$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-ikR\cos\theta}\sin\theta\, dk\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\infty}\frac{1}{ikR}\int_0^{\pi} e^{-ikR\cos\theta}(ikR\sin\theta)\, d\theta\, dk\, d\phi$$

$$= \left(\int_0^{2\pi} d\phi\right)\int_0^{\infty}\frac{1}{ikR}\left[e^{-ikR\cos\theta}\right]_0^{\pi} dk$$

$$= 2\pi\int_0^{\infty}\frac{e^{ikR}-e^{-ikR}}{ikR\,2i}\cdot 2i\, dk$$

$$= \pi\int_0^{\infty}\frac{\sin kR}{kR}\, dk = \frac{4\pi}{R}\int_0^{\infty}\frac{\sin\lambda}{\lambda}\, d\lambda$$

$$= \frac{4\pi}{R}\left(\frac{\pi}{2}\right) = \frac{2\pi^2}{R}$$

From $(*)$

$$u(r) = -\frac{1}{8\pi^3}\int_V\left(\frac{2\pi^2}{R}\right)\rho(r')\, d^3r'$$

$$\Rightarrow u(x,y,z) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, dx'\, dy'\, dz'$$

VARIATIONAL METHODS
OR
CALCULUS OF VARIATION

Functional :-

A function whose domain is a set C_1 of functions and whose range is contained in another set C_2 of functions, not necessarily distinct C_1's is called functional.

According to some authors a functional is a function whose domain is the set of functions and whose range is a subset of real numbers.

STATIONARY VALUE :-

The maximum or minimum value of a function or functional is called stationary value.

Extremal :-

The curve $y=f(x)$ along which a ~~curve~~ functional takes the stationary value is called the extremal.

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2010, 2011, 2016, 2015

Fundamental Theorem Of Variational Calculus.
(one independent Variable)

Statement:-

If $F(x)$ is continuous in the interval (x_1, x_2) and the integral

$$\int_{x_1}^{x_2} F(x)\, G(x)\, dx = 0$$

Where $G(x)$ satisfies the conditions (i) it is an arbitrary function with continuous derivatives in the interval (x_1, x_2) (ii) $G(x_1) = G(x_2) = 0$

then $F(x) \equiv 0 \quad \forall x \in [x_1, x_2]$

PROOF:-

Suppose $F(x) \neq 0$ in $[x_1, x_2]$. Then there is a point $x_0 \in [x_1, x_2]$ s.t. $F(x_0) \neq 0$. Without any loss of generality suppose $F(x_0) > 0$. Then by the continuity of F, there is an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, $\varepsilon > 0$, s.t. $F(x) > 0 \quad \forall x \in [x_0 - \varepsilon, x_0 + \varepsilon]$

Since $G(x)$ is arbitrary, so we define

$$G(x) = \begin{cases} (x - x_0 - \varepsilon)^2 (x - x_0 + \varepsilon)^2 & \text{if } x_0 - \varepsilon < x < x_0 + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

then $$\int_{x_1}^{x_2} F(x)\, G(x)\, dx = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} F(x) (x - x_0 - \varepsilon)^2 (x - x_0 + \varepsilon)^2 dx > 0$$

A contradiction, because $\int_{x_1}^{x_2} F(x) G(x) dx = 0$

So our supposition is wrong and hence $F(x) = 0 \quad \forall x \in [x_1, x_2]$

$$0 = \int_{x_1}^{x_2} F(x)\, G(x)\, dx = \int_{x_1}^{x_0 - \epsilon} F(x) G(x) dx + \int_{x_0 - \epsilon}^{x_0 + \epsilon} F(x) G(x) dx + \int_{x_0 + \epsilon}^{x_2} F(x) G(x) dx$$

$$= 0 + \int_{x_0 - \epsilon}^{x_0 + \epsilon} F(x) G(x) dx + 0 = \int_{x_0 - \epsilon}^{x_0 + \epsilon} F(x) (x - x_0 - \epsilon)^2 (x - x_0 + \epsilon)^2 dx > 0$$

$0 > 0$ contradiction

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2010, 2011, 2016

Fundamental Theorem of Variational Calculus.

(Two independent Variables)

Statement:-

Let a function $F(x,y)$ be continuous in a region D of the xy-plane and $G(x,y)$ be an arbitrary function with continuous partial derivatives in D and let $G(x,y)$ vanishes on the boundary curve C of the domain D.

If
$$\iint_D F(x,y)\,G(x,y)\,dx\,dy = 0$$
Then $F(x,y)=0$ for all (x,y) in the domain D.

Proof:- Suppose given is not true i.e. $F(x,y)\neq 0$ in D. Then there is at least one point (x_0,y_0) in D s.t. $F(x_0,y_0)\neq 0$, Without any loss of generality suppose $F(x_0,y_0)>0$. Since F is continuous in D, so there exists a circular domain centered at (x_0,y_0) and with radius $\varepsilon>0$ i.e. $C_0: (x-x_0)^2+(y-y_0)^2\leq \varepsilon^2$ s.t. $F(x,y)>0$ in this domain. Now as $G(x,y)$ is arbitrary so we can choose

$$G(x,y)=\begin{cases} K[(x-x_0)^2+(y-y_0)^2] & (x,y)\in C_0,\ K>0 \\ 0 & \text{otherwise}\end{cases}$$

Then $\iint_D F(x,y)\,G(x,y)\,dx\,dy = \iint_D F(x,y)\cdot K[(x-x_0)^2+(y-y_0)^2]\,dx\,dy$

A contradiction. So our supposition >0 is wrong. Hence $F(x,y)=0$ $\forall (x,y)\in D$.

Euler's-Lagrange Equation.

THEOREM:-

Let $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

where $y = y(x)$ is a continuous fun. having continuous first and 2nd order derivatives satisfying the following end-point conditions

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2$$

If F is supposed to have cont. 1st and 2nd order derivatives w.r.t. its arguments, then the function $y(x)$ will extremise the given integral if it satisfies the diff. Eq.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\partial F/\partial y'\right) = 0$$

PROOF:-

Clearly b/w two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ infinite no. of curves can be drawn. Let the family of such curves be defined as :-

$$y(x, \alpha) = y(x, 0) + \alpha\, \eta(x) \longrightarrow ①$$

Where $\eta(x)$ denotes the deviation from the curve $y = y(x) = y(x, 0)$ i.e. $\eta(x_1) = 0 = \eta(x_2)$ and α is the parameter labelling different paths and is ~~di~~ indep. of x. We have to find the curve which gives stationary value. We suppose that

the extremal curve corresponds to the value $\alpha = 0$

Now $\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha} dx$

$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \alpha} \right] dx$

Now, $\frac{\partial y}{\partial \alpha} = \eta(x)$ by ①

$\frac{\partial y'}{\partial \alpha} = \eta'(x)$ $\because y'(x,\alpha) = y'(x,0) + \alpha \eta'(x)$

So $\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$

$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \eta(x) dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x) dx \longrightarrow ②$

Consider $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x) dx = \frac{\partial F}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \eta(x) dx$

$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \eta(x) dx$

So $\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right) \eta(x) dx$ by ②

Since $\eta(x)$ satisfies all the conditions of Fundamental Theorem of Variational Calculus so for extreme value

$\frac{\partial I}{\partial \alpha} = 0 \Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial x'}\right) \right) \eta(x) dx = 0$

gives $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial x'}\right) = 0$

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Special Cases.

Case I:- If F is independent of y then

$$\frac{\partial F}{\partial y} = 0 \implies 0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{\partial F}{\partial y'} = \text{constant.}$$

Case II.

If F is independent of x explicitly, then

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$$

$$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right) y' = y'\frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)$$

$$\implies \left(\frac{\partial F}{\partial y}\right) dy = y'\, d\left(\frac{\partial F}{\partial y'}\right) \longrightarrow (1)$$

Now as

$$F = F(y, y')$$

So by differential calculus

$$dF = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy'$$

$$= y'\, d\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'} dy' \quad \text{by (1)}$$

$$\implies dF = d\left(y' \frac{\partial F}{\partial y'}\right)$$

$$\implies d\left(F - y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\implies F - y' \frac{\partial F}{\partial y'} = c$$

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Exp. 1: Find Euler Lagrange Eq. for the following

(i) $F = x^2y^2 - y'^2$

(ii) $F = \sqrt{xy} + y'^2$

(iii) $F = \sin(xy')$

(iv) $F = \dfrac{x^2y'}{\sqrt{1+y'^2}}$

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Sol.

a) $F = x^2y^2 - y'^2$

$\frac{\partial F}{\partial y} = 2x^2y \qquad \frac{\partial F}{\partial y'} = -2y'$

E.L. Eq. is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow 2x^2y - \frac{d}{dx}(-2y') = 0$$

$$\Rightarrow 2x^2y + 2y'' = 0$$

$$\Rightarrow x^2y + y'' = 0$$

$$\Rightarrow y'' + x^2y = 0$$

b)

Here $F = \sqrt{xy} + y'^2$

$$\frac{\partial F}{\partial y} = \frac{1}{2\sqrt{xy}} \cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}$$

$$\frac{\partial F}{\partial y'} = 2y'$$

So $\frac{1}{2}\sqrt{\frac{x}{y}} - \frac{d}{dx}(2y') = 0$

$$\Rightarrow \frac{1}{2}\sqrt{\frac{x}{y}} - 2y'' = 0$$

$$\Rightarrow \sqrt{\frac{x}{y}} - 4y'' = 0$$

$$\Rightarrow \sqrt{x} - 4\sqrt{y}\,y'' = 0$$

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c)

Here

$$F = \sin(xy')$$

As F is indep. of y, so we have

$$\frac{\partial F}{\partial y'} = \text{Constat}$$

$$\frac{\partial F}{\partial y'} = \cos(xy') \cdot x$$

So E. L. Eq. is $x\cos(xy') = C$

d) Here

$$F = \frac{x^2 y'}{\sqrt{1+y'^2}}$$

$$\frac{\partial F}{\partial y'} = \frac{\sqrt{1+y'^2} \cdot x^2 - x^2 y' \left(\frac{y'}{\sqrt{1+y'^2}}\right)}{1+y'^2}$$

$$= \frac{x^2 + x^2 y'^2 - x^2 y'^2}{(1+y'^2)^{3/2}}$$

$$= \frac{x^2}{(1+y'^2)^{3/2}}$$

E. L. Eqn is

$$\frac{\partial F}{\partial y'} = C$$

$$\Rightarrow \frac{x^2}{(1+y'^2)^{3/2}} = C$$

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2010, 2014

Exp. 2: (Brachistochrone / Bernoulli Problem)

(The problem solved by Newton, Bernoulli, and Euler in 1696)

Find the eq. of the path in space down which a particle will fall from one point to another in the shortest possible time.

Sol: Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two points. Then the time taken by the particle is given by

$$\int_A^B dt = \int_A^B \frac{dt}{ds} \cdot ds$$

$$= \int_A^B \frac{1}{v} \sqrt{1+y'^2}\, dx \qquad \because v = \frac{ds}{dt}$$

$$= \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$$

$$= \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

Here $F = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$

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Here F is independent of x explicitly, so Euler's Lagrange eq. is

$$F - y' \frac{\partial F}{\partial y'} = C$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{y}}$$

So $\dfrac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \dfrac{y'}{\sqrt{1+y'^2}\sqrt{y}} = C$

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$$\Rightarrow \frac{1}{y(1+y'^2)} = c^2$$

$$\Rightarrow y' = \sqrt{\frac{1-c^2y}{c^2y}}$$

$$\Rightarrow \sqrt{\frac{c^2y}{1-c^2y}}\,dy = dx$$

Put $c^2y = \sin^2\theta/_2$

$$c^2 dy = 2\cdot\frac{1}{2}\sin\theta/_2 \cos\theta/_2 \, d\theta$$

$$dy = \frac{1}{c^2}\sin\theta/_2 \cos\theta/_2 \, d\theta$$

$$\Rightarrow \frac{\sin\theta/_2}{\cos\theta/_2}\cdot\frac{1}{c^2}\sin\theta/_2 \cos\theta/_2 \, d\theta = dx$$

$$\Rightarrow \frac{1}{c^2}\sin^2\theta/_2 \, d\theta = dx$$

$$\Rightarrow \frac{1}{c^2}\cdot\frac{1-\cos\theta}{2}\,d\theta = dx$$

$$\Rightarrow \theta - \sin\theta = 2c^2x + a$$

$$\Rightarrow x = \frac{1}{2c^2}(\theta - \sin\theta) + a' \rightarrow (1)$$

Also $y = \frac{1}{c^2}\sin^2\theta/_2$

$$\Rightarrow y = \frac{1}{2c^2}(1-\cos\theta) \rightarrow (2)$$

(1) & (2) shows that the req. path is cycloid.

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Exp. 3: Find the curve joining the points (x_1, y_1) and (x_2, y_2) which gives minimum area of surface of revolution generated around (i) y-axis (ii) x-axis.

Proof:

CASE I: About x-axis.

$$\text{Area} = \int_A^B 2\pi y \, ds = \int_A^B 2\pi y \sqrt{1+y'^2} \, dx$$

Here $F = y\sqrt{1+y'^2}$

Euler's eq. is

$$F - y' \frac{\partial F}{\partial y'} = C$$

$$y\sqrt{1+y'^2} - y' \cdot \frac{y y'}{\sqrt{1+y'^2}} = C$$

$$y(1+y'^2) - y y'^2 = C\sqrt{1+y'^2}$$

$$y = C\sqrt{1+y'^2}$$

$$y^2 = C^2(1+y'^2)$$

$$y'^2 = \frac{y^2 - C^2}{C^2}$$

$$y' = \frac{\sqrt{y^2 - C^2}}{C} \Rightarrow \frac{1}{\sqrt{y^2 - C^2}} dy = \frac{1}{C} dx$$

$$\Rightarrow \frac{1}{C} x = \text{Cosh}^{-1}\left(\frac{y}{C}\right) + a$$

$$\Rightarrow x = C \, \text{Cosh}^{-1}\left(\frac{y}{C}\right) + b \qquad b = aC$$

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Case II :- About Y-axis.

$$A = \int_A^B 2\pi x \, ds = \int_A^B 2\pi x \sqrt{1+y'^2} \, dy$$

Here $F = x\sqrt{1+y'^2}$

Euler's Eq. is

$$\frac{\partial F}{\partial y'} = c$$

$$\frac{xy'}{\sqrt{1+y'^2}} = c$$

$$\frac{x^2 y'^2}{1+y'^2} = c^2$$

$$\Rightarrow \quad y' = \frac{c}{\sqrt{x^2 - c^2}}$$

$$\Rightarrow \quad dy = \frac{c}{\sqrt{x^2-c^2}} dx$$

$$\Rightarrow \quad y = c \cos\left(\frac{x}{c}\right) + a$$

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Exp. 4:

Give the geometrical inter-pretation of the variational calculus

$$\int_0^1 \sqrt{1+y'^2}\,dx$$

with the boundary conditions

$y(0)=0, \quad y(1)=1$

Solve the problem for the extremal. Find the sationary value of the integral and compare it with the values of the integral which are obtained for the curves that join the same end points but are different from the extremal.

Solution.

Geometrical Interpreation:

Given that

$$I = \int_0^1 \sqrt{1+y'^2}\,dx$$

$$= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{(dx)^2+(dy)^2} = \int_0^1 ds$$

Which gives the distance b/w two points on a surface. As the points are $A=(0,0)$ and $B=(1,1)$. So this shows that the surface is a plane, xy-plane. We have to find the value of y which minimize this integral.

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Now we find y

Here

$$F = \sqrt{1+y'^2}$$

Since F is indep. of y

So Euler's Lagrange eq. is

$$\frac{\partial F}{\partial y'} = c$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow y' = \sqrt{\frac{c^2}{1-c^2}} = a$$

$$\Rightarrow y = ax+b$$

$y(0) = 0 \Rightarrow b = 0$

$y(1) = 1 \Rightarrow a(1) = 1 \Rightarrow a = 1$

So $y(x) = x$

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STATIONARY VALUE:-

$$S.V. = \int_0^1 \sqrt{1+y'^2}\,dx$$

$$= \int_0^1 \sqrt{1+1^2}\,dx = \sqrt{2}\,(x)\Big|_0^1$$

$$= \sqrt{2}\,(1-0)$$

$$= \sqrt{2}$$

Comparison:

Choose $y = x^2$, then

$$I = \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$$

$$= \int_0^1 \sqrt{1+(2x)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{2x\sqrt{1+4x^2}}{2} + \frac{1}{2}\ln\left(\frac{2x+\sqrt{1+4x^2}}{1}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right]$$

$$= \frac{1}{2}\left[2.24 + .72\right] = 1.48 > \sqrt{2}$$

So $\sqrt{2}$ is the minimum i.e. stationary value.

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20/11/2012

Theorem:-

State suitable boundary conditions for the functional

$$I = \int F(x, y, y', y'', y''', \dots, y^{(n)})\, dx$$

and find Euler-Lagrange for the corresponding extremal.

Sol.

Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two fixed points in plane throug which the extremal curve is situated.

Then the boundary conditions will be $y(x_1) = y'(x_1) = y''(x_1) = \dots = y^{(n)}(x_1) = \text{constant}$

$y(x_2) = y'(x_2) = y''(x_2) = \dots = y^{(n)}(x_2) = \text{Constant}$

Let δy be the variation in y and let δF, δI be the corresponding variations in F & I respectively.

Then

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' + \dots + \frac{\partial F}{\partial y^n} \delta y^n \right) dx \longrightarrow (1)$$

Consider

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta y' dx = \frac{\partial F}{\partial y'} \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y\, dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y\, dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = \frac{\partial F}{\partial y''} \delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \delta y' dx$$

$$= 0 - \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y\, dx \right]$$

$$= (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y\, dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^n} \delta y^n dx = (-1)^n \int_{x_1}^{x_2} \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) \delta y\, dx$$

So by ①

$$\delta I=\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\delta y+(-1)^1\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y+(-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\delta y\right]dx$$

$$\delta I=\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\right]\delta y\,dx$$

For the extremal curve $\delta I=0$

$$\Rightarrow\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\right]\delta y\,dx=0$$

From Fundamental Theorem

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right)+\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)=0$$

$\because \delta y$ is arbitrary

So req. Euler-Lagrange Eqn is

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\cdots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)=0$$

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Theorem:-

Find the extremal of the functional

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n')\, dx$$

with b.cs.

$$y_k(x_1) = \text{Constant} \qquad y_k(x_2) = \text{Constant}$$
$$k = 1, 2, \dots, n$$

Sol.

Let δI be the variation in I corresponding to the variation δy in y. Then

$$\delta I = \int_{x_1}^{x_2} \left[\left\{ \frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_2}\delta y_2 + \dots + \frac{\partial F}{\partial y_n}\delta y_n \right\} + \left\{ \frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2'}\delta y_2' + \dots + \frac{\partial F}{\partial y_n'}\delta y_n' \right\} \right] dx \rightarrow (*)$$

Now consider $\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k'} \delta y_k' \right) dx$

$$= \frac{\partial F}{\partial y_k'} \delta y_k \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) \delta y_k \, dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) \delta y_k \, dx \qquad k = 1, 2, 3, \dots, n$$

$$\Rightarrow \delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'} \right) \right) \delta y_1 + \left(\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'} \right) \right) \delta y_2 + \dots + \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_n'} \right) \right) \delta y_n \right] dx$$

For extremal $\delta I = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k \, dx = 0 \qquad k = 1, 2, \dots, n$$

As δy_k is arbitrary so

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) = 0 \qquad k = 1, 2, \dots, n$$

Which are reqd. E.L. Eqs.

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2012, 2013

Exp. 1:-

Find the extremal for

$$I = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx$$

with $y(0) = 0$ $\quad y(\pi/2) = 1$

$z(0) = 0$ $\quad z(\pi/2) = -1$

Sol.

Here two variable y & z are functions of x. So corresponding E. L. Eqns. are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial z}{\partial z'}\right) = 0 \longrightarrow (2)$$

Here $F = y'^2 + z'^2 + 2yz$

$\frac{\partial F}{\partial y} = 2z$ $\quad \frac{\partial F}{\partial z} = 2y$

$\frac{\partial F}{\partial y'} = 2y'$ $\quad \frac{\partial F}{\partial z'} = 2z'$

By (1) $2z - \frac{d}{dx}(2y') = 0$

$\Rightarrow z - y'' = 0 \longrightarrow (3)$

By (2) $2y - \frac{d}{dx}(2z') = 0 \Rightarrow y - z'' = 0 \rightarrow (4)$

From (3) & (4) $z - z'''' = 0 \Rightarrow z'''' - z = 0$

$\Rightarrow (D^4 - 1)z = 0$

A. E. is $D^4 - 1 = 0$

$\Rightarrow D = \pm 1, \pm i$

$z = Ae^x + Be^{-x} + C\cos x + D\sin x$

$z(0) = 0 \Rightarrow A + B + C = 0 \rightarrow (5)$

$z(\pi/2) = -1 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} + D = -1 \rightarrow (6)$

Again from (3) & (4)

~~$y - y'''' = 0 \Rightarrow y'''' - y = 0$~~

~~$\Rightarrow (D^4 - 1)y = 0$~~

~~$\Rightarrow D = \pm 1, \pm i$~~

~~$\Rightarrow y = Ae^{x} + Be^{-x} + C\cos x + D\sin x$~~

~~$y(0) = 0 \Rightarrow A + B$~~

Also $z' = Ae^{x} - Be^{-x} - C\sin x + D\cos x$

$z'' = Ae^{x} + Be^{-x} - C\cos x - D\sin x$

$\Rightarrow$ As $y = z'' \Rightarrow y = Ae^{x} + Be^{-x} - C\cos x - D\sin x$

$y(0) = 0 \Rightarrow A + B - C = 0 \longrightarrow (7)$

$y(\pi/2) = 1$

$\Rightarrow Ae^{\pi/2} + Be^{-\pi/2} - D = 1 \longrightarrow (8)$

Solving (5), (6), (7), (8)

$A = 0, \quad B = 0, \quad C = 0, \quad D = -1$

So $y = \sin x \qquad z = \cos x$

Exp. 2:- Find the extremal 2012, 2013, 2016, 2014

$$I = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\,dx$$

with $y(0) = 1, \; y(\pi/2) = 0, \; y'(0) = 0, \; y'(\pi/2) = 1$

Sol.

Here $F = x^2 - y^2 + y''^2$

and corresponding E-L Eq. is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = -2y \qquad \frac{\partial F}{\partial y'} = 0 \qquad \frac{\partial F}{\partial y''} = 2y''$$

So by (1)

$$-2y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$$

$\Rightarrow -2y + 2y'''' = 0 \Rightarrow y'''' - y = 0$

A.E. is $D^4 - 1 = 0$

$\Rightarrow D = \pm 1, \pm i$

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$\Rightarrow y(x) = Ae^{x} + Be^{-x} + C\cos x + D\sin x$

$y'(x) = Ae^{x} - Be^{-x} - C\sin x + D\cos x$

$y(0) = 0 \Rightarrow A + B + C = 0 \rightarrow (2)$

$y(\pi/2) = 0 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} + D = 0 \rightarrow (3)$

$y'(0) = 0 \Rightarrow A - B + D = 0 \rightarrow (4)$

$y'(\pi/2) = 1 \Rightarrow Ae^{\pi/2} - Be^{-\pi/2} - C = 0 \rightarrow (5)$

Solving (2), (3), (4), (5)

$A = \frac{1}{2}(1 + e^{-\pi/2}) \qquad B = \frac{1}{2}(1 - e^{\pi/2})$

$C = \frac{1}{2}(e^{\pi/2} - e^{-\pi/2}) = \sinh \pi/2$

$D = -\frac{1}{2}(e^{\pi/2} + e^{-\pi/2}) = -\cosh \pi/2$

Hence

$y = \frac{1}{2}(1 + e^{-\pi/2})e^{x} + \frac{1}{2}(1 - e^{\pi/2})e^{-x}$

$+ \sinh \pi/2 \cos x - \cosh \pi/2 \sin x$

Exp. 3. Use C.V. to prove that a st. line is the shortest distance b/w two points in a plane.

Sol. Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be two points in xy-plane. Then we have to minimize

$$I = \int_A^B \sqrt{1 + y'^2}\, dx$$

with $y(x_1) = y_1, \quad y(x_2) = y_2$

Here $F = \sqrt{1 + y'^2}$

Solving $\frac{\partial F}{\partial y'} = c$ we get

$$y = ax + b$$

Which is a st. line.

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E.L.Eq. For Two Independent Variables in The Integral Theorem.

The extremal of the functional

$$I = \iint_R F(x, y, u, u_x, u_y)\,dx\,dy$$

Where u has different values inside the region R but is prescribed on the boundary of the region R. Moreover u is supposed to possess continuous partial derivatives upto and order, is given by

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

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Sol.

$$\delta I = \iint_R \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right) dx\,dy$$

Now consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\delta u_x$

$$\Rightarrow \frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Similarly

$$\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$$

So

$$\delta I = \iint_R \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\right] dx\,dy$$

$$= \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\,dx\,dy$$

$$+ \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right] dx\,dy$$

$\Rightarrow \delta I = I_1 + I_2$

Where

$$I_1 = \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\,dx\,dy$$

$$I_2 = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right)\right]dx\,dy$$

Applying Green's Theorem on I_2

$$\iint_R \left[\frac{\partial}{\partial x}\left(\overset{Q}{\frac{\partial F}{\partial u_x}\delta u}\right) + \frac{\partial}{\partial y}\left(\overset{P}{\frac{\partial F}{\partial u_y}\delta u}\right)\right]dx\,dy = \int_C \left(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx\right)\delta u$$

$$\left(\iint_R (Q_x - P_y)\,dx\,dy = \int_C (P\,dx + Q\,dy)\right)$$

Since u is prescribed on the boundary so $\delta u = 0$ $\Rightarrow I_2 = 0$

$$\Rightarrow \delta I = \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\,dx\,dy$$

For extremal

$$\delta I = 0$$

$$\Rightarrow \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\,dx\,dy = 0$$

As δu is arbitrary, so for Fundamental Theorem of calculus of variation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

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Theorem:-

Given the functional

$$I = \iiint_R F(x, y, z; u, u_x, u_y, u_z)\, dx\, dy\, dz$$

Where u has different values in a three dimensional region R but is prescribed on the boundary surface S of the region; it is assumed that u has continuous partial derivatives upto 2nd order in the region R. Then necessary condition for this function to have an extremum is that $u(x,y,z)$ must satisfy the PDE

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

Proof:-

Given I implies

$$\delta I = \iiint_V \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z\right] dx\, dy\, dz \quad \text{(1)}$$

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Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\delta u_x$

$$\Rightarrow \frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Sim

$$\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$$

$$\frac{\partial F}{\partial u_z}\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$$

So by (1)

$$\delta I = \iiint_V \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right] dx\,dy\,dz$$

$$- \iiint_V \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right] dx\,dy\,dz$$

Consider

$$I_1 = \iiint_V \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right] dx\,dy\,dz$$

$$= \iiint_V \operatorname{div}\left(\frac{\partial F}{\partial u_x}\delta u\,\hat{i} + \frac{\partial F}{\partial u_y}\delta u\,\hat{j} + \frac{\partial F}{\partial u_z}\hat{k}\right) dx\,dy\,dz$$

$$= \iiint_V \operatorname{div} G\, dV$$

where $G = \left(\frac{\partial F}{\partial u_x}\hat{i} + \frac{\partial F}{\partial u_y}\hat{j} + \frac{\partial F}{\partial u_z}\hat{k}\right)\delta u$

& $dV = dx\,dy\,dz$

Now by Gauss's Divergence Theorem

$$\iiint_V \operatorname{div} G\, dV = \iint_S (G\cdot n)\,ds$$

Where n is outward normal to the region. But since $\delta u = 0$ on the boundary surface S, so $I_1 = 0$.

$$\Rightarrow \delta I = \iiint_V \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right]\delta u\, dx\,dy\,dz = 0$$

For extremal $\delta I = 0$

But as δu is arbitrary. So by F.T.C.o.V.

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

Constrained Extrema

These problems are also called variational problems with constraints or variational problems with side conditions or isoperimetrical problems.

In such problems we have to find a curve $y = y(x)$ which extremize the integral

$$I = \int F \, dx$$

subject to the certain condition.

Working Rule.

The curve $y = y(x)$ extremize the integral

$$I = \int F \, dx$$

subject to the condition

G = Constant.

Then y satisfies the E.L. Eq.

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

Where

$H = F + \lambda G$, λ is any parameter

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2014

Exp. 1. Find the curve joining the points A(0,0) & B(1,0) with the given length s.t. the y-coordinate of its centroid is minimum.

Sol. Let $y = y(x)$ be the req. curve. Then y-coordinate of its centroid is given by

$$\bar{y} = \frac{\int_0^1 xy\,ds}{\int_0^1 x\,ds} = \frac{\int_0^1 y\,ds}{\int_0^1 ds}$$

$$\Rightarrow \bar{y} = \frac{\int_0^1 y\sqrt{1+y'^2}\,dx}{l}$$

Where $l = \int_0^1 \sqrt{1+y'^2}\,dx$ is the length of the curve through the given points. Thus we have to find the curve $y = y(x)$ which minimizes $\bar{y}$ subject to the condition

$$l = \int_0^1 \sqrt{1+y'^2}\,dx$$

This is an isoperimetric problem with

$$F = y\sqrt{1+y'^2} \qquad G = \sqrt{1+y'^2}$$

Let $H = F + \lambda G$

$$= y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}$$

& corresponding E-L. Eq. be

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0.$$

$$\frac{\partial H}{\partial y} = \sqrt{1+y'^2}$$

$$\frac{\partial H}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}} + \frac{\lambda y'}{\sqrt{1+y'^2}}$$

So $\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{(\lambda+y)y'}{\sqrt{1+y'^2}}\right) = 0$

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But this will lead to the complication. As H is indep. of x, so we we use

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$\Rightarrow (y+\lambda)\sqrt{1+y'^2} - y'\left(\frac{(\lambda+y)y'}{\sqrt{1+y'^2}}\right) = c$$

$$\Rightarrow (y+\lambda)(1+y'^2) - (\lambda+y)y'^2 = c\sqrt{1+y'^2}$$

$$\Rightarrow y + yy'^2 + \lambda + \lambda y'^2 - \lambda y'^2 - yy'^2 = c\sqrt{1+y'^2}$$

$$\Rightarrow (y+\lambda)^2 = c^2(1+y'^2)$$

$$\Rightarrow \frac{(y+\lambda)^2 - c^2}{c^2} = y'^2$$

$$\Rightarrow y' = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c}$$

$$\Rightarrow \frac{c}{\sqrt{(y+\lambda)^2 - c^2}}\,dy = dx$$

$$\Rightarrow c\cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x + a$$

$y(0) = 0 \Rightarrow c\cosh^{-1}\left(\frac{0+\lambda}{c}\right) = 0 + a$

$$\Rightarrow \cosh^{-1}\left(\frac{\lambda}{c}\right) = \frac{a}{c}$$

$y(1) = 0 \Rightarrow c\cosh^{-1}\left(\frac{0+\lambda}{c}\right) = 1 + a$

$$\Rightarrow \cosh^{-1}\left(\frac{\lambda}{c}\right) = \frac{1}{c} + \frac{a}{c}$$

Now $\frac{\lambda}{c} = \cosh(a/c)$ & $\lambda/c = \cosh\left(\frac{a}{c} + \frac{1}{c}\right)$

$$\Rightarrow \cosh(-a/c) = \cosh\left(\frac{a}{c} + \frac{1}{c}\right)$$

$$\Rightarrow -a/c = \frac{a}{c} + \frac{1}{c}$$

$$\Rightarrow -\frac{2a}{c} = \frac{1}{c} \Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow c\cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x - \frac{1}{2} = \frac{2x-1}{2}$$

$$\Rightarrow \frac{y+\lambda}{c} = \cosh\left(\frac{2x-1}{2c}\right)$$

$$\Rightarrow y = c\left[\cosh\left(\frac{2x-1}{2c}\right) - \frac{\lambda}{c}\right]$$

$$\Rightarrow y = c\left[\cosh\left(\frac{2x-1}{2c}\right) - \cosh\left(\frac{a}{c}\right)\right]$$

$$\Rightarrow y = c\left[\cos\left(\frac{2x-1}{2c}\right) - \cosh\left(\frac{1}{2c}\right)\right]$$

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2010, 2011, 2016

Exp 2: Show that a solid of revolution which for a given surface area has maximum volume is a sphere.

Or

Find a curve which generates a surface of revolution of a given area which encloses maximum vol.

Solution.

Let $y = y(x)$ with $y(0) = 0$, $y(a) = 0$ be rotated about x-axis so as to generate a surface of revolution. An element of surface area is $2\pi y\,ds$, where $ds = \sqrt{(dx)^2 + (dy)^2}$. Then the total area will be

$$A = 2\pi \int_0^a y\sqrt{1+y'^2}\,dx$$

Also in this case

$$V = \pi \int_0^a y^2\,dx$$

Here we have to minimize V subject to A.

$$\Rightarrow F = y^2 \qquad G = y\sqrt{1+y'^2}$$

$$\Rightarrow H = F + \lambda G \Rightarrow H = y^2 + \lambda y\sqrt{1+y'^2}$$

Since H is indep. of x, so E.L. Eq. is

$$H - y'(\partial H/\partial y') = c$$

$$y^2 + \lambda y\sqrt{1+y'^2} - y'\left(\frac{\lambda y y'}{\sqrt{1+y'^2}}\right) = c$$

$$y^2\sqrt{1+y'^2} + \lambda y + \lambda y y'^2 - \lambda y y'^2 = c\sqrt{1+y'^2}$$

$$\frac{\lambda y}{c - y^2} = \sqrt{1+y'^2}$$

$$\Rightarrow \frac{\lambda^2 y^2}{(c-y^2)^2} = 1+y'^2 \Rightarrow y'^2 = \frac{\lambda^2 y^2 - (c-y^2)^2}{(c-y^2)^2}$$

$$\Rightarrow y' = \frac{\sqrt{\lambda^2 y^2 - (c-y^2)^2}}{c-y^2}$$

Now using $y(0)=0$

$$\Rightarrow y'(0) = \frac{\sqrt{0-c^2}}{c-0} \longrightarrow ?$$

$$\Rightarrow y'(0) = \frac{\sqrt{-c^2}}{c}$$

$$\Rightarrow c\, y'(0) = \sqrt{-c^2}$$

$\Rightarrow c = 0$ $\because$ $y'(0)$ is finite and y is real.

So $y' = \dfrac{\sqrt{\lambda^2 y^2 - y^4}}{-y^2}$

$$\Rightarrow y' = -\frac{\sqrt{\lambda^2 - y^2}}{y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{\lambda^2 - y^2}}{y}$$

$$\Rightarrow \frac{y}{\sqrt{\lambda^2 - y^2}}\,dy = -dx$$

$$\Rightarrow -\frac{1}{2}\int (\lambda^2 - y^2)^{-1/2}(-2y)\,dy = -\int dx$$

$$\Rightarrow \sqrt{\lambda^2 - y^2} = x + b$$

$$\Rightarrow \lambda^2 - y^2 = (x+b)^2$$

$$\Rightarrow (x+b)^2 + y^2 = \lambda^2$$

Which is eqn. of a sphere with center $(-b, 0)$ and radius λ. Which is reqd. proof.

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Exp 3. (Dido's Problem)

Find the closed curve of given length which encloses maximum area.

Solution:

Clearly the required curve is convex. Also we note that a st. line that bisects a closed curve bounding a maximum area will divide the area in equal halves.

We suppose that X-axis is the line which divides the curve in two equal halves and it meets the X-axis at $A(x_1, 0)$ and $B(x_2, 0)$ as shown in the figure. Then

Area enclosed $= A_1 = I = \int_A^B y\,dx$

$$\Rightarrow I = \int_A^B y\,dx$$

Also length of the curve $= l = \int_A^B \sqrt{1+y'^2}\,dx$

we have to maximize I subject to l.

Here $F = y$ $\quad G = \sqrt{1+y'^2}$

$$H = y + \lambda\sqrt{1+y'^2}$$

E. L. Eq. is $H - y'\dfrac{\partial H}{\partial y'} = c$

$$\Rightarrow y + \lambda\sqrt{1+y'^2} - y'\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = c$$

$$y\sqrt{1+y'^2} + \lambda + \lambda y'^2 - \lambda y'^2 = c\sqrt{1+y'^2}$$

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$$\Rightarrow (c-y)\sqrt{1+y'^2} = \lambda$$

$$\sqrt{1+y'^2} = \frac{\lambda}{c-y}$$

$$\Rightarrow y'^2 = \frac{\lambda^2}{(c-y)^2} - 1 = \frac{\lambda^2-(c-y)^2}{(c-y)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\lambda^2-(c-y)^2}}{c-y}$$

$$\Rightarrow \int \frac{c-y}{\sqrt{\lambda^2-(c-y)^2}}\,dy = \int dx$$

$$\Rightarrow \int [\lambda^2-(c-y)^2]^{-\frac{1}{2}}(c-y)\,dy = \int dx$$

$$\Rightarrow \frac{1}{2}\int [\lambda^2-(c-y)^2]^{-\frac{1}{2}}\,2(c-y)\,dy = \int dx$$

$$[\lambda^2-(c-y)^2]^{\frac{1}{2}} = x+d$$

$$\Rightarrow \lambda^2-(c-y)^2 = (x+d)^2$$

$$\Rightarrow (x+d)^2+(y-c)^2 = \lambda^2$$

$$\Rightarrow (x-\lambda_1)^2+(y-\lambda_2)^2 = \lambda^2$$

Which is the eq. of a circular arc. So required closed curve is a circle.

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Geodesic Problems

A geodesic is a curve of shortest length joining two points in space.

Example 1:

Find the arc of shortest length b/w two given points in a plane, using polar coordinates (r, θ).

Sol. Let A & B be two points in plane. Then arc length b/w A & B is given by

$$I = \int_A^B ds = \int_A^B \sqrt{(dr)^2 + (r\,d\theta)^2}$$

$$= \int_A^B \sqrt{r^2 + r'^2}\, d\theta$$

Subject to $r(\theta_1) = \text{Constant}$

and $r(\theta_2) = \text{Constant}$

Here $F(\theta, r, r') = \sqrt{r^2 + r'^2}$

Since F is independent of θ explicitly, so E.L. Eq. is

$$F - r'\frac{\partial F}{\partial r'} = C$$

$$\sqrt{r^2 + r'^2} - r'\frac{r'}{\sqrt{r^2 + r'^2}} = C$$

$$r^2 + r'^2 - r'^2 = C\sqrt{r^2 + r'^2}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r}{c}\sqrt{r^2 - c^2}$$

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$$\Rightarrow \int \frac{c\,dr}{r\sqrt{r^2-c^2}} = \int d\theta$$

$$\Rightarrow \sec^{-1}\left(\frac{r}{c}\right) = \theta + \alpha$$

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$$\Rightarrow \frac{r}{c} = \sec(\theta + \alpha)$$

$$\Rightarrow c = r\cos(\theta + \alpha)$$

$$= r[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$$

$$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

$$\Rightarrow c = x\cos\alpha - y\sin\alpha$$

$$\Rightarrow ax + by + c = 0, \text{ Where } a = -\cos\alpha,\ b = \sin\alpha$$

Which is eq. of line.

Hence curve of shortest length in a plane is a straight line.

Example 2:- 2015

Find a curve of shortest length on the surface of a sphere.

Solution:-

Let A and B be two points on the surface of a sphere of radius a. Then arch length b/w A and B is given by

$$I = \int_A^B ds = \int_A^B \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2}$$

As $r = a$. So $dr = 0$

$$\Rightarrow I = \int_A^B \sqrt{a^2 d\theta^2 + a^2\sin^2\theta\, d\varphi^2}$$

$$= \int_A^B a\sqrt{1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2}\, d\theta$$

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Here we have to minimize it.

Now

$$F(\theta,\varphi) = a\sqrt{1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2} = a\sqrt{1+\sin^2\theta(\varphi')^2}$$

E. L. Eq. is

$$\frac{\partial F}{\partial \varphi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \varphi'}\right) = 0 - \frac{d}{d\theta}\left(\frac{\sin^2\theta\,\varphi'}{\sqrt{1+\sin^2\theta\,\varphi'^2}}\right) = 0$$

$$\Rightarrow \frac{\sin^2\theta\,\varphi'}{\sqrt{1+\sin^2\theta(\varphi')^2}} = c_1$$

$$\Rightarrow \frac{\sin^4\theta\,\varphi'^2}{1+\sin^2\theta\,\varphi'^2} = c_1^2$$

$$\Rightarrow \varphi' = \frac{c_1}{\sin\theta\sqrt{\sin^2\theta - c_1^2}}$$

$$\Rightarrow \int d\varphi = \int \frac{c_1}{\sin\theta\sqrt{\sin^2\theta - c_1^2}}\,d\theta$$

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$$\Rightarrow \phi = \int \frac{c_1}{\sin\theta\sqrt{\sin^2\theta - c_1^2}}\,d\theta$$

$$\Rightarrow \phi = \int \frac{c_1}{\frac{1}{\operatorname{cosec}\theta}\sqrt{\frac{1}{\operatorname{cosec}^2\theta} - c_1^2}}\,d\theta$$

$$= \int \frac{c_1 \operatorname{cosec}^2\theta}{\sqrt{1 - c_1^2\operatorname{cosec}^2\theta}}\,d\theta$$

$$= \int \frac{c_1 \operatorname{cosec}^2\theta}{\sqrt{1 - c_1^2(1+\cot^2\theta)}}\,d\theta$$

Put $\cot\theta = \psi$

$-\operatorname{cosec}^2\theta\, d\theta = d\psi$

$\Rightarrow \operatorname{cosec}^2\theta\, d\theta = -d\psi$

$$\Rightarrow \phi = c_1 \int \frac{-d\psi}{\sqrt{1 - c_1^2(1+\psi^2)}}$$

$$= -\int \frac{d\psi}{\sqrt{1 - c_1^2 - c_1^2\psi^2}}$$

$$= -\frac{c_1}{c_1}\int \frac{d\psi}{\sqrt{\frac{1-c_1^2}{c_1^2} - \psi^2}}$$

$$= -\frac{c_1}{c_1}\int \frac{d\psi}{\sqrt{c_2^2 - \psi^2}} \qquad c_2 = \frac{1-c_1^2}{c_1^2}$$

$$= \frac{c_1}{c_1}\int \frac{-d\psi}{\sqrt{c_2^2 - \psi^2}}$$

$$\Rightarrow \phi = \cos^{-1}(\psi/c_2) + c_3$$

$$\Rightarrow \phi = \cos^{-1}\left(\frac{\cot\theta}{c_2}\right) + c_3$$

$$\Rightarrow \cos(\phi - c_3) = \frac{\cot\theta}{c_2}$$

$$\Rightarrow c_2[\cos\phi \cos c_3 + \sin\phi \sin c_3] = \cot\theta$$

$$\Rightarrow (c_2\cos c_3)\cos\phi + (c_2\sin c_3)\sin\phi = \frac{\cos\theta}{\sin\theta}$$

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$\Rightarrow \alpha \sin\theta \cos\varphi + \beta \sin\theta \sin\varphi = \cos\theta$

$\Rightarrow \alpha (a \sin\theta \cos\varphi) + \beta (a \sin\theta \sin\varphi) = a \cos\theta$

$\Rightarrow \alpha x + \beta y = z$

Which is the eq. of the plane through the center of the sphere. Hence the curve of shortest length joining A & B is the arc of the great circle through A and B.

Example 3:-

Find the geodesic curve for the cylinder

$$x^2 + y^2 = a^2$$

Sol.

$$I = \int_A^B \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$$

$r = a \quad dr = 0$

$$\Rightarrow I = \int_A^B \sqrt{a^2 d\theta^2 + dz^2}$$

$$= \int_A^B \sqrt{a^2 + z'^2}\, d\theta$$

E. L. Eq. is $\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0$

$$0 - \frac{d}{d\theta}\left(\frac{z'}{\sqrt{a^2 + z'^2}}\right) = 0$$

$$z' / \sqrt{a^2 + z'^2} = c_1$$

$$\Rightarrow z' = c_1 \sqrt{a^2 + z'^2}$$

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$$\Rightarrow z'^2 = c_1^2 a^2 + c_1^2 z'^2$$

$$\Rightarrow (1 - c_1^2) z'^2 = c_1^2 a^2$$

$$\Rightarrow z'^2 = c_1^2 a^2 / 1 - c_1^2 = \alpha^2$$

$$\Rightarrow z' = \alpha$$

$$\Rightarrow \frac{dz}{d\theta} = \alpha$$

$$\Rightarrow z = \alpha \theta + \beta$$

$$\Rightarrow z = \alpha \cdot \tan^{-1}\left(\frac{y}{x}\right) + \beta$$

$$\Rightarrow \frac{z - \beta}{\alpha} = \tan^{-1}\frac{y}{x}$$

$$\Rightarrow y = x \tan\left(\frac{z - \beta}{\alpha}\right)$$

Which is required curve.

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Exp. 4:-

Find the shortest distance b/w the points $A(1, -1, 0)$ and $B(2, 1, -1)$ in the plane $15x - 7y + z - 22 = 0$

Sol.

Here $I = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$= \int_A^B \sqrt{1 + (y')^2 + (z')^2}\, dx$$

Here $F = \sqrt{1 + y'^2 + z'^2}$ & $G = 15x - 7y + z - 22$

and E.L. Eqs. are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \longrightarrow (1)$$

$$\& \quad \frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0 \longrightarrow (2)$$

Where $H = \sqrt{1 + y'^2 + z'^2} + \lambda(15x - 7y + z - 22)$

By (1) $-7\lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$

$$\Rightarrow 7\lambda + \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \longrightarrow (3)$$

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By ② $\quad -\lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \longrightarrow ④$

Eliminating λ from ③ & ④

$$-\frac{d}{dx}\left(\frac{y'+7z'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \Rightarrow \frac{y'+7z'}{\sqrt{1+y'^2+z'^2}} = C_1 \longrightarrow *$$

Now given that

$15x - 7y + z - 22 = 0$

$15 - 7y' + z' = 0$

$\Rightarrow z' = 7y' - 15$

So $* \Rightarrow$

$$\frac{y' + 7(7y' - 15)}{\sqrt{1 + y'^2 + (7y' - 15)^2}} = C_1$$

$\Rightarrow 50(250 - C_1^2)y'^2 + 210(C_1^2 - 50)y' + (11025 - 226) = 0$

$\Rightarrow \alpha_1 y'^2 + \beta_1 y' + \gamma_1 = 0$

Which is quadratic in y'. Let α be the one root of this eq. Then $y' = \alpha \Rightarrow y = \alpha x + \beta$

Now $y(1) = -1 \Rightarrow \alpha + \beta = -1$

& $y(2) = 1 \Rightarrow 2\alpha + \beta = 1$

$\alpha = 2$

$\Rightarrow \beta = -3$

Hence $y = 2x - 3$

So $z = 7y - 15x + 22 = -x + 1$

So S.D. $= \int_1^2 \sqrt{1 + (2)^2 + (-1)^2}\, dx = \sqrt{6}$

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Applications To Mechanics

Principle of Least Action.

Let a particle move in an extremal field of force which is conservative. If the motion takes place in the interval of time t_1 to t_2, where $t_2 > t_1$. Then the actual path traced by the particle is the one along which

$$I = \int_{t_1}^{t_2} L\, dt$$

is minimum. Where L is Langrangian and for a conservative system

$$L = K.E. - P.E. = T - V$$

HAMILTON'S PRINCIPLE:-

According to this principle, the path of motion of a rigid body in the time interval $t_2 - t_1$ is such that the integral

$$I = \int_{t_1}^{t_2} L\, dt$$

has stationary value. Where L is Langrangian.

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Exp. 1: Find the eq. of motion of particle moving in a conservative field of force described by the function $V(x,y,z)$.

Solution:

Here $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$$L = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(x,y,z)$$

By the principle of least action the eqs. of motion are given by

$$I = \int_{t_1}^{t_2} L\,dt$$

Where I has the minimum value, which implies that

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \rightarrow (3)$$

(1) implies

$$-\frac{\partial V}{\partial x} - \frac{d}{dt}(m\dot{x}) = 0$$

$$\Rightarrow \frac{\partial V}{\partial x} + m\ddot{x} = 0 \rightarrow (4)$$

(2) implies that $\frac{\partial V}{\partial y} + m\ddot{y} = 0 \rightarrow (5)$

(3) implies that $\frac{\partial V}{\partial z} + m\ddot{z} = 0 \rightarrow (6)$

(4), (5), (6) are required eqs. of motion.

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Exp. 2:

Use the principle of least action to obtain DE describing the vibrations of S. H. oscilator

Sol.

By definition for a particle executing S.H.M. along X-axis, the force F is given by

$$F = -kx$$

Where x denotes displacement from center of vibration; The constant k which is +ve, is called spring constant. The K.E. is given by

~~F = -Kx~~

$$T = \frac{1}{2} m\dot{x}^2$$

As $F = dV/dx$

So $V = -\int kx\, dx$

$$= -\frac{1}{2} kx^2$$

So $L = T - V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$

So

E.L. Eq. is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\Rightarrow kx - \frac{d}{dt}(m\dot{x}) = 0$$

$$\Rightarrow kx - m\ddot{x} = 0$$

$$\Rightarrow m\ddot{x} - kx = 0$$

Exp. 3:.

Obtain the eq. of motion of a stretched string by invoking Hamilton's Principle of least action.

Sol. Let us consider a string of uniform density ρ and length l s.t. initially it is stretched along X-axis, and when plucked let it vibrate in XY-plane. Consider the K.E. of an element ds of the string. Then

$$dT = \frac{1}{2} dm v^2 = \frac{1}{2} \rho y_t^2 \, ds$$

$$\Rightarrow \quad T = \frac{1}{2} \rho \int_0^l y_t^2 \, ds$$

Here we assume that deflection of the string is very small. So

$$T = \frac{1}{2} \rho \int_0^l y_t^2 \, dx$$

and

P.E. = Increase in length due to tension of the string = Work done by the tension in producing extension $ds - dx$

$$= \int_0^l (ds - dx) T$$

$$= T\left[\int_0^l \left[1 + \frac{1}{2} y_x^2\right] dx - l\right]$$

$$= T\left[l + \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx - l\right]$$

$$= \frac{T}{2}\int_0^l y_x^2 \, dx$$

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So $L = T - V = \frac{\rho}{2}\int_0^l y_t^2\,dx - \frac{T}{2}\int_0^l y_x^2\,dx$

$= \int_0^l (\rho y_t^2 - T y_x^2)\,dx$

By Hamilton's principle the eq. of motion of the string will correspond to stationary value of $\int_{t_1}^{t_2} L\,dt$ i.e. $\frac{1}{2}\int_{t_1}^{t_2}\int_0^l [\rho y_t^2 - T y_x^2]\,dx\,dt$

Let $F = \rho y_t^2 - T y_x^2$

For the external

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) = 0 \rightarrow (1)$$

$\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial y_x} = -2T y_x$ $\frac{\partial F}{\partial y_t} = +2\rho y_t$

So by (1)

$$0 - \frac{\partial}{\partial x}(-2T y_x) - \frac{\partial}{\partial t}(+2\rho y_t) = 0$$

$$\Rightarrow \cancel{2}T \frac{\partial^2 y}{\partial x^2} - \cancel{2}\rho \frac{\partial^2 y}{\partial t^2} = 0$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = 0$$

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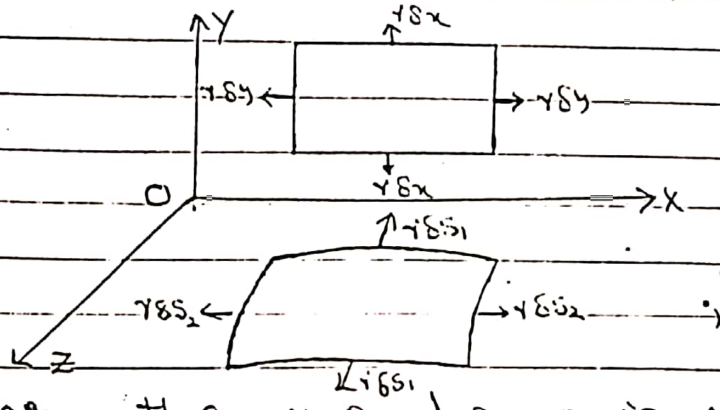
Example: ?

Discuss the vibrations of a stretched membrane and obtain its eq. of motion by using H.P.

Sol.

We consider the transverse vibrations of a membrane of arbitrary shape which is supposed to be perfectly flexible but inextensible. First we calculate K.E. and P.E. of the vibrating membrane.

Let the membrane lies in XY-plane. As in case of the string, we suppose that displacements are small and are ⊥ to XY-plane i.e. in the direction of Z-axis.



When the membrane is vibrating, a rectangular element of area ABCD originally lying in XY-plane is brought to the position of curved element ds in space centered around the point (x,y,z) at time t. Here the corresponding z-coordinate denotes the displacement of the membrane at time t, i.e. $z = z(x,y,t)$

To calculate P.E and K.E. we analyse the motion of an element of the membrane

$$T = \frac{1}{2}\int z_t^2 \, dm$$

$$= \frac{1}{2}\int z_t^2 \rho \, ds = \frac{\rho}{2}\int z_t^2 \, dx dy$$

Where we have taken $ds = dxdy$ in the 1st approximation

$$\because ds = \sqrt{1 + z_x^2 + z_y^2}\, dxdy$$

and z_x, z_y are neglecting because vibrations are very small.

To calculate Potentional energy V of the vibrating membrane, we have to calculate the work done in beginning membrane from original position in XY-plane to its current position.

First we calculate the work done on the rectangular element ABCD of the area $dxdy$ in briging it to the position of element δS.

Now if T is the tension per unit length in the membrane then two forces each of magnitude Tdx are pulling the sides AD and BC, whereas two forces each of magnitude Tdy are pulling the sides AB and CD of the element ABCD.

Now remembering that the work done on these sides are respectively $(ds_1 - dx)\,Tdy$ and $(ds_2 - dy)\,Tdx$

Now work done for the element of area ds will be

$$dV = T(ds_1 - dx)dy + T(ds_2 - dy)dx$$
$$= T(ds_1 dy + ds_2 dx - 2dx dy)$$

Now

$$ds_1 dy = \sqrt{(dx)^2 + (dz)^2}\, dy = \sqrt{1 + z_x^2}\, dx dy$$
$$= (1 + \tfrac{1}{2} z_x^2) dx dy$$
$$ds_2 dx = \sqrt{(dy)^2 + (dz)^2}\, dx = (1 + \tfrac{1}{2} z_y^2) dx dy$$

Where we have used the fact that z_x and z_y are small because of displacements being small. Therefore we can write

$$dV = T\left[(1 + z_x^2) dx dy + (1 + \tfrac{1}{2} z_y^2) dx dy - 2 dx dy\right]$$
$$= T/2\, (z_x^2 + z_y^2) dx dy$$

Hence

$$V = T/2 \int_S (z_x^2 + z_y^2) dx dy$$

Hence $L = T - V = \frac{1}{2}\int_S (\rho z_t^2 - T(z_x^2 + z_y^2)) dx dy$

By Hamilton's Principle :

$$\frac{1}{2}\int_{t_1}^{t_2}\int_S (\rho z_t^2 - T z_x^2 - T z_y^2) dx dy dt = 0$$

Here $F = F(t, x, y, z, z_t, z_x, z_y) = \rho z_t^2 - T z_x^2 - T z_y^2$ is and corresponding E-L Eq^{n} is

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial z_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

$$\Rightarrow 0 - \frac{\partial}{\partial t}(2\rho z_t) - \frac{\partial}{\partial x}(-2T z_x) - \frac{\partial}{\partial y}(-2T z_y) = 0$$

$$\Rightarrow -\rho z_{tt} + T z_{xx} + T z_{yy} = 0$$

$$\Rightarrow z_{xx} + z_{yy} = \frac{\rho}{T} z_{tt} \qquad c = \sqrt{T/\rho}$$

$$\Rightarrow \text{Req. is } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\, \partial^2 z/\partial t^2$$

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بسم More Examples

Exp. 1:- If φ and ψ are two functions s.t. $\delta\varphi(x_1) = \delta\varphi(x_2) = 0$

$\delta\psi(x_1) = 0 = \delta\psi(x_2)$

Then show that the necessary condition for

$$I = \int_{x_1}^{x_2} F(x, \phi, \psi, \phi', \psi')\,dx$$

to have a stationary is that

$$\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right) = 0$$

$$\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

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Sol.

Given $I = \int_{x_1}^{x_2} F(x, \varphi, \psi, \varphi', \psi')\,dx$

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \varphi}\delta\varphi + \frac{\partial F}{\partial \psi}\delta\psi + \frac{\partial F}{\partial \varphi'}\delta\varphi' + \frac{\partial F}{\partial \psi'}\delta\psi'\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \varphi}\delta\varphi - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\delta\varphi + \frac{\partial F}{\partial \psi}\delta\psi - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\delta\psi\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\right]\delta\varphi\,dx + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right]\delta\psi\,dx$$

For external $\delta I = 0$ (stationary value)

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right)\right]\delta\varphi\,dx + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right]\delta\psi\,dx = 0$$

Since $\delta\varphi$ and $\delta\psi$ are arbitrary

So
$$\frac{\partial F}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \varphi'}\right) = 0$$

and
$$\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

Exp. 2:

Solve that the E.L. Eqs. for the functional

$$I = \int_A^B F(x, y, y', z, z')\, dx$$

admit that

(i) $\frac{\partial F}{\partial y'} = c$, if F is indep. of y

(ii) $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = c$, if F is indep. of x.

Sol.

Corresponding E.L. Eqs. are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \longrightarrow (2)$$

(i) If F is indep. of y, then

$$\frac{\partial F}{\partial y} = 0$$

So then $0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = \int 0 \cdot dx$$

$$\Rightarrow \frac{\partial F}{\partial y'} = C$$

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(ii) Here $F = F(y, y', z, z')$

$$\Rightarrow dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial z'}dz'$$

Also by (1)

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{\partial F}{\partial y}dy = y'\, d(\partial F/\partial y')$$

Similarly $\frac{\partial F}{\partial z}dz = z'\, d(\partial F/\partial z')$

Hence

$$dF = y' d(\partial F/\partial y') + \partial F/\partial y'\, dy' + z' d(\partial F/\partial z') + \partial F/\partial z'\, dz'$$

$$= d\left(y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'}\right)$$

$$\Rightarrow F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = c$$

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Exp. 3. Find eigenvalues and eigenfunctions of the functional

$$I = \int_0^3 \left[(2x+3)^2 y'^2 - y^2\right] dx$$

subject to conditions $y(0) = 0 = y(3)$ and $\int_0^3 y^2(x)\,dx = 1$

Sol.

Here $F = (2x+3)^2 y'^2 - y^2$

& $G = y^2$

So $H = F + \lambda G = (2x+3)^2 y'^2 - y^2 + \lambda y^2$

The corresponding E.L. Eq. is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left\{\frac{\partial H}{\partial y'}\right\} = 0$$

$$\Rightarrow -2y + 2\lambda y - \frac{d}{dx}\left\{2(2x+3)^2 y'\right\} = 0$$

$$\Rightarrow \frac{d}{dx}\left\{2(2x+3)^2 y'\right\} + 2y - 2\lambda y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(2x+3)^2 y'\right\} + (1-\lambda)y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(2x+3)^2 \frac{dy}{dx}\right\} + (1)\,y + (-1)\lambda y = 0$$

Which is an S.L. Eq. with

$p(x) = (2x+3)^2$, $q(x) = 1$, $r(x) = -1$

The above eq. can be written as

$$(2x+3)^2 \frac{d^2y}{dx^2} + 4(2x+3)\frac{dy}{dx} + (1-\lambda)y = 0 \rightarrow (1)$$

Which is Cauchy Euler's eq.

So put $2x+3 = e^t \Rightarrow \ln(2x+3) = t$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}e^t$$

$$\Rightarrow \frac{dy}{dx} = 2e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = 4e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

So eq. (1) becomes

$$4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+(1-\lambda)y=0$$

A.E. is

$$4D^2+4D+(1-\lambda)=0$$

$$\Rightarrow D=\frac{-4\pm\sqrt{16-4(1-\lambda)(4)}}{2(4)}$$

$$=-\frac{1\pm\sqrt{\lambda}}{2}$$

If $\lambda\geq 0$ then solution is trivial.
If $\lambda<0$, then we can write $\lambda=-\lambda_0,\ \lambda_0>0$
Then general solution is given by

$$y=e^{-t/2}\left(c_1\cos\sqrt{\lambda_0}\,\tfrac{t}{2}+c_2\sin\sqrt{\lambda_0}\,\tfrac{t}{2}\right)$$

$$\Rightarrow y(x)=(2x+3)^{-1/2}\left[c_1\cos\left(\tfrac{1}{2}\sqrt{\lambda_0}\ln(2x+3)\right)+c_2\sin\left(\tfrac{1}{2}\sqrt{\lambda_0}\ln(2x+3)\right)\right]$$

Applying $y(0)=0$, $y(3)=0$

$$\frac{c_1}{\sqrt{3}}\cos\left(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right)+\frac{c_2}{\sqrt{3}}\sin\left(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right)=0 \quad \text{(2)}$$

and $\frac{c_1}{3}\cos(\sqrt{\lambda_0}\ln 3)+\frac{c_2}{3}\sin(\sqrt{\lambda_0}\ln 3)=0$

For non trivial sol.

$$\begin{vmatrix}\cos(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3) & \sin(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3)\\ \cos(\sqrt{\lambda_0}\ln 3) & \sin(\sqrt{\lambda_0}\ln 3)\end{vmatrix}=0$$

$$\Rightarrow \cos(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3)\sin(\sqrt{\lambda_0}\ln 3)-\sin(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3)\cos(\sqrt{\lambda_0}\ln 3)=0$$

$$\sin\left(\sqrt{\lambda_0}\ln 3-\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right)=0$$

$$\Rightarrow \sin\left(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right)=0$$

$$\Rightarrow \tfrac{1}{2}\sqrt{\lambda_0}\ln 3=n\pi \qquad n=\pm1,\pm2,\pm3,\ldots$$

$$\Rightarrow \sqrt{\lambda_0}=\frac{2n\pi}{\ln 3}$$

$$\Rightarrow \lambda_{0n}=\frac{4n^2\pi^2}{(\ln 3)^2} \qquad n=1,2,3,\ldots$$

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Which are required eigen values.
Now by ②

$$C_1 \cos n\pi + C_2 \sin n\pi = 0$$
$$\Rightarrow C_1 = 0$$

Hence the corresponding Eigen-functions are given by

$$y_n(x) = \frac{A_n}{\sqrt{2x+3}} \sin\left[\frac{n\pi \ln(2x+3)}{\ln 3}\right] \quad n = 1, 2, 3$$

Where the constants D_n can be determined from the condition

$$\int_0^3 y^2(x)\, dx = 1$$

which gives

$$D_n = \pm 2/\sqrt{\ln 3}$$

✓ **Exp. 4:-** On what curves can the functional $I = \int_0^{\pi/2} (y'^2 - y^2)\, dx$

$y(0) = 0$, $y(\pi/2) = 1$ be extremised.

Sol.

Here $F = y'^2 - y^2$

E.L. Eq. is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$-2y - \frac{d}{dx}(2y') = 0$$
$$\Rightarrow y'' + y = 0$$

A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$\Rightarrow y = A \cos x + B \sin x$$

Using given conditions

$$A = 0 \quad B = 1$$

So $y = \sin x$.

Exp. 5: The Lagrangian L for a system of n particles is a function of generalized coordinates q_i'. Show the minimization of the integral

$$I = \int_{t_1}^{t_2} L\, dt$$

leads to the following eq. of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) = 0$$

Sol. Here $L = L(q_i, q_i')$

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial q_i'}\delta q_i'\right) dt$$

Now

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i'}\delta q_i'\, dt = \frac{\partial L}{\partial q_i'}\delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\delta q_i\, dt$$

$$= 0 - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\delta q_i\, dt$$

$\because q_i$ are fixed at t_1 and t_2

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\right)\delta q_i\, dt$$

For the external curve $\delta I = 0$.

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\right]\delta q_i\, dt = 0$$

Since δq_i is arbitrary, so by fundamental Theorem of V. M.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) = 0 \qquad i = 1, 2, \ldots, n.$$

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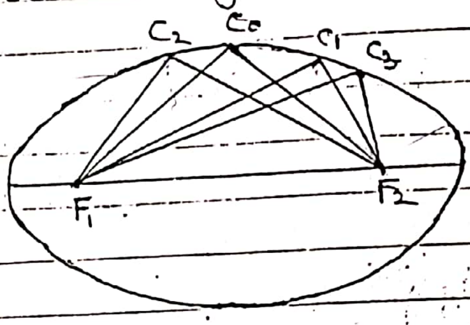
Exp. 6:-

Use Law of reciprocity to prove the isoscelles triangle has the smallest perimeter for a given area and a given base.

Sol.

First we show using a well known property of an ellipse that of all triangles with a given base and a given perimeter, the isosceles Δ has maximum area.



Let F_1, F_2 be foci of an ellipse. We take F_1F_2 as the base for the triangles $C_1F_1F_2$, $C_2F_1F_2$, $C_3F_1F_2$ etc.

Since the sum of distances of any point on an ellipse from its foci is a constant. It follows that the triangles $C_1F_1F_2$, $C_2F_1F_2$ etc. have the same perimeter. However the triangle $C_0F_1F_2$ which has the largest altitude and which is an isosceles triangle, has the maximum area.

By Law of Reciprocity, it follows that all triangles with given base and given area the isosceles triangles have the smallest perimeter.

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Question: Find the shortest distance b/w $A(1,0,-1)$ and $B(0,-1,1)$ lying on the surface $x+y+z=0$

Solution: Here we have to minimize $I=\int_A^B \sqrt{(dx)^2+(dy)^2+(dz)^2}$

$$\Rightarrow I=\int_A^B \sqrt{1+(y')^2+(z')^2}\,dx$$

subject to the condition $x+y+z=0$

So here $F=\sqrt{1+(y')^2+(z')^2}$

& $G=x+y+z$

with $y(0)=-1$ $\quad z(0)=1$

& $y(1)=0$ $\quad , z(1)=-1$

Set $H=F+\lambda G=\sqrt{1+(y')^2+(z')^2}+\lambda(x+y+z)$

Corresponding E.L. Equations are

$$\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\partial H/\partial y'\right)=0 \quad \rightarrow ①$$

$$\& \quad \frac{\partial H}{\partial z}-\frac{d}{dx}\left(\partial H/\partial z'\right)=0 \quad \rightarrow ②$$

From ① & ② respectively we get

$$\lambda-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right)=0 \quad \rightarrow ③$$

and $\lambda-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right)=0 \rightarrow ④$

subtracting ③ from ④

$$\frac{d}{dx}\left(\frac{y'-z'}{\sqrt{1+y'^2+z'^2}}\right)=0$$

$$\Rightarrow \frac{y'-z'}{\sqrt{1+y'^2+z'^2}}=C_1$$

Now $x+y+z=0$

$\Rightarrow 1+y'+z'=0$

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$\Rightarrow z' = -1 - y'$ So

$$\frac{y' - (-1-y')}{\sqrt{1+y'^2+(-1-y')^2}} = C_1$$

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$$\Rightarrow \frac{1+2y'}{\sqrt{1+y'^2+1+y'^2+2y'}} = C_1$$

$$\Rightarrow \frac{1+2y'}{\sqrt{2+2y'^2+2y'}} = C_1$$

$$1+4y'^2+4y' = C_1^2(2+2y'+2y'^2)$$

$$\Rightarrow 1+4y'^2+4y'-2C_1^2-2C_1^2y'-2C_1^2y'^2 = 0$$

$$\Rightarrow (4-2C_1^2)y'^2+(4-2C_1^2)y'+(1-2C_1^2) = 0$$

$$ay'^2+by'+c = 0 \rightarrow (5)$$

where $a = 4-2C_1^2$, $b = 4-2C_1^2$, $c = 1-2C_1^2$

is quadratic in y' and can be made to have real roots. Let its one of real roots be α, then

$$y' = \alpha$$

$$\Rightarrow y = \alpha x + \beta$$

Using B.Cs. we have $y = x-1$ and $z = -2x+1$

So

$$I = \int_A^B \sqrt{1+y'^2+z'^2}\, dx$$

$$= \int_0^1 \sqrt{1+1^2+4}\, dx = \int_0^1 \sqrt{6}\, dx$$

$$= \sqrt{6}\, x\Big|_0^1 = \sqrt{6}(1-0)$$

$$= \sqrt{6}$$

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ORDINARY DIFFERENTIAL EQUATIONS :-

BESSEL'S DIFFERENTIAL EQUATION:-

A second order linear differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$ is known

as BESSEL'S differential equation.
Solution of this differential equation is oftenly denoted by $J_n(x)$ and is known as Bessel's function. This equation is also sometimes known as Bessel's differential equation of order n, where 'n' is a number.

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SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION:-

Bessel's differential equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \longrightarrow (1)$$

put $y=\frac{u(x)}{\sqrt{x}}$

Then $\frac{dy}{dx}=\frac{\sqrt{x}\cdot\frac{d}{dx}u(x)-u(x)\cdot\frac{1}{2\sqrt{x}}}{x}$

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$$x\frac{dy}{dx} = \sqrt{x}\frac{du(x)}{dx} - \frac{u(x)}{2\sqrt{x}}$$

$$x\frac{dy}{dx} = \frac{2x\,u'(x) - u(x)}{2\sqrt{x}}$$

Again differentiating w.r.t 'x'

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{\sqrt{x}\,[2x\,u''(x) + 2u'(x) - u'(x)] - [2x\,u'(x) - u(x)]\cdot 1/2\sqrt{x}}{2x}$$

$$x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{2}\left[2x\sqrt{x}\,u''(x) + \sqrt{x}\,u'(x)\right] - \left[\sqrt{x}\,u'(x) - \frac{u(x)}{2\sqrt{x}}\right]$$

$$= \frac{1}{2}\left[2x\sqrt{x}\,u''(x) + \sqrt{x}\,u'(x) - \sqrt{x}\,u'(x) + \frac{u(x)}{2\sqrt{x}}\right]$$

Using these values, equ (1) becomes

$$x\sqrt{x}\,u''(x) + \frac{u(x)}{4\sqrt{x}} + (x^2 - n^2)\frac{u(x)}{\sqrt{x}} = 0$$

Dividing by $x\sqrt{x}$

$$\Rightarrow u''(x) + \frac{u(x)}{4x^2} + \frac{(x^2-n^2)\,u(x)}{x^2} = 0$$

$$u''(x) + \left[\frac{1}{4x^2} + \frac{(x^2-n^2)}{x^2}\right]u(x) = 0$$

$$u''(x) + \left[\frac{1}{4x^2} + 1 - \frac{n^2}{x^2}\right]u(x) = 0$$

$$u''(x) + \left[1 - \frac{n^2 - 1/4}{x^2}\right]u(x) = 0$$

for special case when $n = \pm 1/2$, then

$n^2 - 1/4 = 0$

$$\Rightarrow u''(x) + [1 - 0]\,u(x) = 0$$

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Characteristic equation is

$D^2 + 1 = 0 \quad \Rightarrow \quad D = \pm i$

$\Rightarrow u(x) = c_1 \cos x + c_2 \sin x$

$\Rightarrow y(x) = \dfrac{c_1 \cos x + c_2 \sin x}{\sqrt{x}}$ is a general

Solution of Bessel's differential equation for $n = \pm 1/2$.

SERIES SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION:-

Let us consider Bessel's differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \longrightarrow (1)$$

, where n is non -ve integer

Let $y = \sum_{s=0}^{\infty} C_s x^{r+s}$

$$\frac{dy}{dx} = \sum_{s=0}^{\infty} (r+s) C_s x^{r+s-1}.$$

$$\frac{d^2y}{dx^2} = \sum_{s=0}^{\infty} (r+s)(r+s-1) C_s x^{r+s-2}.$$

putting all these values in (1)

$$x^2\left[\sum_{s=0}^{\infty} C_s (r+s)(r+s-1) x^{r+s-2}\right] + x\left[\sum_{s=0}^{\infty} (r+s) C_s x^{r+s-1}\right]$$

$$+ (x^2 - n^2) \sum_{s=0}^{\infty} C_s x^{r+s} = 0$$

$$\sum_{s=0}^{\infty} C_s (r+s)(r+s-1) x^{r+s} + \sum_{s=0}^{\infty} (r+s) C_s x^{r+s} + \sum_{s=0}^{\infty} C_s x^{r+s+2}$$

$$- n^2 \sum_{s=0}^{\infty} C_s x^{r+s} = 0$$

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$$\sum_{s=0}^{\infty} C_s[(r+s)(r+s-1)+(r+s)-n^2]x^{r+s} + \sum_{s=0}^{\infty} C_s x^{r+s+2} = 0$$

$$\sum_{s=0}^{\infty} C_s[(r+s)^2-n^2]x^{r+s} + \sum_{s=0}^{\infty} C_s x^{r+s+2} = 0$$

Now, equating coefficients of like powers of 'x'.

Coefficient of x^r : $C_0(r^2-n^2)=0$ ⟶ (2)

" " : x^{r+1} : $C_1[(r+1)^2-n^2]=0$ ⟶ (3)

" " x^{r+2} : $C_2[(r+2)^2-n^2]+C_0=0$ ⟶ (4)

" " x^{r+3} : $C_3[(r+3)^2-n^2]+C_1=0$ ⟶ (5)

" " x^{r+4} : $C_4[(r+4)^2-n^2]+C_2=0$ ⟶ (6)

⋮

" " x^{r+2m} : $C_{2m}[(r+2m)^2-n^2]+C_{2m-2}=0$ ⟶ (7)

" " x^{r+2m+1} : $C_{2m+1}[(r+2m+1)^2-n^2]+C_{2m-1}=0$ ⟶ (8)

from (2)

$C_0(r^2-n^2)=0$

As $C_0 \neq 0$, so $r^2-n^2=0 \Rightarrow r^2=n^2 \Rightarrow r=\pm n$

Since $n \geq 0$, so we consider $r=n$

from (3)

$C_1[(r+1)^2-n^2]=0$

$\Rightarrow C_1[(n+1)^2-n^2]=0 \quad \because r=n$

$\Rightarrow C_1[n^2+2n+1-n^2]=0$

$\Rightarrow C_1[2n+1]=0$

Since here $n \neq -\frac{1}{2} \quad \because n \geq 0$

$\Rightarrow 2n+1 \neq 0 \Rightarrow C_1=0$

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Now from (8)

$$C_{2m+1} = -\frac{C_{2m-1}}{(r+2m+1)^2 - n^2}$$

put $m = 1$.

$$C_3 = -\frac{C_1}{(r+3)^2 - n^2} = 0 \Rightarrow C_3 = 0 \quad (\because C_1 = 0)$$

Put $m=2$

$$C_5 = -\frac{C_3}{(r+5)^2 - n^2} = 0 \Rightarrow C_5 = 0 \quad (\because C_3 = 0)$$

Continuing this way, we obtain

$C_1 = 0,\ C_3 = 0,\ C_5 = 0, \ldots\ldots,\ C_{2m+1} = 0, \ldots$

Now choose $C_0 = \frac{1}{2^n . n!}$

Now from ⑦

$$C_{2m} = -\frac{C_{2m-2}}{(r+2m)^2 - n^2} = -\frac{C_{2m-2}}{(n+2m)^2 - n^2}$$

$$= -\frac{C_{2m-2}}{n^2 + 4m^2 + 4mn - n^2} = -\frac{C_{2m-2}}{4m^2 + 4mn}$$

$$= -\frac{C_{2m-2}}{4m(m+n)}$$

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Put $m = 1$

$$C_2 = -\frac{C_0}{4(n+1)} \quad \because C_0 = \frac{1}{2^n n!}$$

$$= -\frac{1}{2^n . n!\ 4(n+1)}$$

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$$= \frac{-1}{2^n \cdot 2^2 (n+1)!}$$

$$= \frac{-1}{2^{n+2} (n+1)! \cdot 1!}$$

Put $m=2$

$$C_4 = \frac{-C_2}{8(n+2)}$$

$$= \frac{(-1)^2}{2^2 (n+2) \cdot 2 \cdot 2^{n+2} (n+1)!}$$

$$= (-1)^2 \frac{1}{2^{n+4} (n+2)! \cdot 2!}$$

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Put $m=3$

$$C_6 = \frac{-C_4}{12(n+3)}$$

$$= (-1)^3 \cdot \frac{1}{2^{n+6} (n+3)! \cdot 6}$$

$$= (-1)^3 \cdot \frac{1}{2^{n+6} (n+3)! \, 3!}$$

⋮

$$C_{2s} = (-1)^s \cdot \frac{1}{2^{n+2s} (n+s)! \, s!}$$

Now, $y = \sum_{s=0}^{\infty} C_s x^{r+s} = C_0 x^r + C_1 x^{r+1} + C_2 x^{r+2}$

$$+ C_3 x^{r+3} + C_4 x^{r+4} + C_5 x^{r+5} + \cdots\cdots$$

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As $C_1 = 0$, $C_3 = 0$, $C_5 = 0, \ldots\ldots$

$$\Rightarrow y = C_0 x^{r} + C_2 x^{r+2} + C_4 x^{r+4} + \cdots = \sum_{s=0}^{\infty} C_{2s} x^{r+2s}$$

$$= \sum_{s=0}^{\infty} C_{2s} x^{n+2s}$$

$$y = \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{n+2s} (n+s)!\, s!} x^{n+2s}$$

$$y = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

As we write for the sol $y = J_n(x)$

$\Rightarrow$ Bessel's equation is :

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

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QUESTION:-

Prove the following Recurrance relation:

(i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(ii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(differential recurrance formula)

(iii) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

(Pure recurrance formula)

(iv) $x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

(v) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

(vi) $J_0'(x) = -J_1(x)$

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(vii) $J_{-n}(x) = (-1)^n J_n(x)$

(viii) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Solution:-

(i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

As $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$

$$J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (n+2s) \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} n\left(\frac{x}{2}\right)^{n+2s} \cdot \left(\frac{2}{x}\right) \cdot \frac{1}{2} +$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2(s) \left(\frac{x}{2}\right)^{n+2s-1} \frac{1}{2}$$

$$= \frac{n}{x} \sum_{s=0} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} + \sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!(n+s)!} \left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \frac{n}{x} J_n(x) + \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!(n+p+1)!} \left(\frac{x}{2}\right)^{n+2(p+1)-1}$$

(here we put $s = p+1$)

$$= \frac{n}{x} J_n(x) + \sum_{p=0}^{\infty} \frac{(-1)^p \cdot (-1)}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{(n+1)+2p}$$

$$= \frac{n}{x} J_n(x) - \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{n+1+2p}$$

$$= \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$\Rightarrow x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

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(ii) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

As $J_n(x) = \sum_{s=0} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$

Differentiating both sides w.r.t 'x'

$$\Rightarrow J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} (n+2s) \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} (n+s+s) \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} (n+s) \left(\frac{x}{2}\right)^{(n-1)+2s} \cdot \frac{1}{2} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \cdot s \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s-1)!} \left(\frac{x}{2}\right)^{n-1+2s} \cdot \frac{1}{2} + \frac{1}{2} \sum_{s=1} \frac{(-1)^s}{(s-1)!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \frac{1}{2} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,[(n-1)+s]!} \left(\frac{x}{2}\right)^{n-1+2s} + \frac{1}{2} \sum_{s=1} \frac{(-1)^s}{(s-1)!\,(n+s)!} \left(\frac{x}{2}\right)^{n-1+2s}$$

$$= \frac{1}{2} J_{n-1}(x) + \frac{1}{2} \sum_{s=1} \frac{(-1)^s}{(s-1)!\,(n+s)!} \left(\frac{x}{2}\right)^{n-1+2s}$$

put $s = p+1$.

$$\Rightarrow J_n'(x) = \frac{1}{2} J_{n-1}(x) + \frac{1}{2} \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{n+1+2p}$$

$$J_n'(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} \sum_{p=0} \frac{(-1)^p}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{n+1+2p}$$

$$= \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x)$$

$$\Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

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(iii) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Since we know that

$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

$\Rightarrow J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \longrightarrow (i)$

Also $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

$\Rightarrow J_n'(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x) \longrightarrow (ii)$

from (i) and (ii)

$\frac{n}{x} J_n(x) - J_{n+1}(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x)$

$\frac{n}{x} J_n(x) = \frac{1}{2} J_{n-1}(x) + \frac{1}{2} J_{n+1}(x)$

$\Rightarrow \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

(iv) $x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

As $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$

$$J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} (n+2s) \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{-n}{2} + \frac{2n+2s}{2}\right) \left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{-n}{2}\right) \left(\frac{x}{2}\right)^{n+2s-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{2n+2s}{2}\right) \left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \left(\frac{-n}{x}\right) \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s-1} \left(\frac{x}{2}\right) + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,[(n-1)+s]!} \left(\frac{x}{2}\right)^{n-1+2s}$$

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$$J_n'(x) = \left(\frac{-n}{x}\right)\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s} + J_{n-1}(x)$$

$$= \left(\frac{-n}{x}\right) J_n(x) + J_{n-1}(x)$$

$\Rightarrow x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

$\Rightarrow x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

(v) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

$L.H.S = \frac{d}{dx}\left[x^{-n} J_n(x)\right]$

$= x^{-n} J_n'(x) + (-n)\, x^{-n-1} J_n(x) \longrightarrow (i)$

But $J_n'(x) = \left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x)$, put in (i)

$\Rightarrow \frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n}\left[(n/x) J_n(x) - J_{n+1}(x)\right] - n x^{-n-1} J_n(x)$

$= n \cdot x^{-n-1} J_n(x) - x^{-n} J_{n+1}(x) - n x^{-n-1} J_n(x)$

$\Rightarrow \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

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(vi) $J_0'(x) = -J_1(x)$

As we know that

$x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

put $n = 0$, we get

$\Rightarrow x J_0'(x) + 0 = x J_{-1}(x)$

$\Rightarrow J_0'(x) = J_{-1}(x)$

$J_0'(x) = -J_1(x) \quad \because J_{-n}(x) = (-1)^n J_n(x)$

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(vii) $J_{-n}(x) = (-1)^n J_n(x)$.

As $J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(-n+s)!} \left(\frac{x}{2}\right)^{-n+2s}$

put $s = p+n \Rightarrow p = s-n$.

$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{p+n}}{(p+n)!\cdot p!} \left(\frac{x}{2}\right)^{-n+2(p+n)}$

$= \sum_{p=0}^{\infty} \frac{(-1)^p (-1)^n}{p!\,(n+p)!} \left(\frac{x}{2}\right)^{n+2p}$

$= (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p)!} \left(\frac{x}{2}\right)^{n+2p}$

$\Rightarrow J_{-n}(x) = (-1)^n \cdot J_n(x)$

(viii) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

L.H.S $= \frac{d}{dx}\left[x^n J_n(x)\right]$

$= x^n J_n'(x) + n x^{n-1} J_n(x)$

As we know that

$J_n'(x) = \left(\frac{-n}{x}\right) J_n(x) + J_{n-1}(x)$

$\Rightarrow \frac{d}{dx}\left[x^n J_n(x)\right] = x^n \left[\left(\frac{-n}{x}\right) J_n(x) + J_{n-1}(x)\right] + n x^{n-1} J_n(x)$

$= -n x^{n-1} J_n(x) + x^n J_{n-1}(x) + n x^{n-1} J_n(x)$

$= x^n J_{n-1}(x)$

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QUESTION:-

Show that

$$e^{\frac{x}{2}[t-1/t]} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Sol

L.H.S:-

$$e^{\frac{x}{2}[t-1/t]} = e^{\frac{x}{2}[t-t^{-1}]}$$

$$= e^{\frac{x}{2}t} \cdot e^{-\frac{x}{2}t^{-1}}$$

$$= \sum_{r=0}^{\infty} \frac{(xt/2)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{(-xt^{-1}/2)^s}{s!}$$

$$\because e^x = \sum_{s=0}^{\infty} \frac{x^s}{s!}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{r+s} (t)^{r-s} (-1)^s \frac{1}{r!\,s!}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,r!}\, t^{r-s} \left(\frac{x}{2}\right)^{r+s}$$

put $r-s=n$, then

$$e^{\frac{x}{2}[t-1/t]} = \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\, t^n \left(\frac{x}{2}\right)^{n+2s}$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}\right] t^n$$

$$e^{\frac{x}{2}[t-1/t]} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

QUESTION:-

Prove the Jacobi series

(i) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\cdots$

(ii) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\cdots$

(iii) $\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\cdots$

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(iv) $\sin(x\cos\theta) = 2\{J_1(x)\cos\theta - J_3(x)\cos 3\theta + \ldots\ldots\}$

Solution:-

We know that

$$e^{x/2[t-1/t]} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$= J_0(x)t^0 + J_{-1}(x)t^{-1} + J_1(x)t^1 + J_{-2}(x)t^{-2} + J_2(x)t^2 + J_{-3}(x)t^{-3} + J_3(x)t^3 + \ldots$$

Now $J_{-n}(x) = (-1)^n J_n(x)$, so

$$e^{x/2[t-1/t]} = J_0(x) - J_1(x)t^{-1} + J_1(x)t^1 + J_2(x)t^{-2} + J_2(x)t^2 - J_3(x)t^{-3} + J_3(x)t^3 + J_4(x)t^{-4} + J_4(x)t^4 + \ldots$$

$$= J_0(x) + [t^1 - t^{-1}]J_1(x) + [t^2 + t^{-2}]J_2(x) + [t^3 - t^{-3}]J_3(x) + [t^4 + t^{-4}]J_4(x) + \ldots$$

put $t = e^{i\theta}$, then

$$e^{x/2[e^{i\theta} - e^{-i\theta}]} = J_0(x) + (e^{i\theta} - e^{-i\theta})J_1(x) + (e^{2i\theta} + e^{-2i\theta})J_2(x) + (e^{3i\theta} - e^{-3i\theta})J_3(x) + (e^{4i\theta} + e^{-4i\theta})J_4(x) + \ldots$$

$$e^{ix\sin\theta} = J_0(x) + i\,2\sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 2i\sin 3\theta\, J_3(x) + 2\cos 4\theta\, J_4(x) + \ldots$$

$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = (J_0(x) + 2\cos 2\theta J_2(x) + 2\cos 4\theta J_4(x) + \ldots) + i(2\sin\theta J_1(x) + 2\sin 3\theta J_3(x) + \ldots)$$

comparing real and imaginary parts

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta J_2(x) + 2J_4(x)\cos 4\theta + \ldots$$

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \ldots$$

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(iii) $\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + \cdots$

(iv) $\sin(x\cos\theta) = 2\{J_1(x)\cos\theta - J_3(x)\cos 3\theta + \cdots\}$

Sol:

As $\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2J_1(x)\sin\theta \cdot i + 2J_2(x)\cos 2\theta + 2J_3(x)\sin 3\theta \cdot i + 2J_4(x)\cos 4\theta + \cdots$

Replacing θ by $\pi/2 - \theta$

$\cos(x\sin(\pi/2-\theta)) + i\sin(x\sin(\pi/2-\theta)) = J_0(x) + 2J_1(x) \cdot \sin(\pi/2-\theta) \cdot i + 2J_2(x)\cos 2(\pi/2-\theta) + 2J_3(x)\sin 3(\pi/2-\theta) \cdot i + 2J_4(x)\cos 4(\pi/2-\theta) + \cdots$

$\Rightarrow \cos(x\cos\theta) + i\sin(x\cos\theta) = J_0(x) + 2J_1(x)\cos\theta \cdot i - 2J_2(x)\cos 2\theta - 2J_3(x)\cos 3\theta \cdot i + 2J_4(x) \cdot \cos 4\theta + \cdots$

Comparing real and imaginary parts.

$\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$

$\sin(x\cos\theta) = 2J_1(x)\cos\theta - 2J_3(x)\cos 3\theta + \cdots$

EXPRESSION FOR $J_n(x)$, where n is a half of an odd integer.

QUESTION:- Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

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Solution:-

We know that

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

Now, Define

$$\Gamma(x) = \int_0^{\infty} e^{-t} \cdot t^x \, dt \quad , \text{then}$$

If x is a non-negative integer then

$\Gamma(x) = x!$ & $\Gamma(x) = x\Gamma(x-1)$

and $\Gamma(1/2) = \int_0^{\infty} e^{-t} \cdot t^{1/2} \, dt$

put $t = \theta^2$; then $dt = 2\theta \, d\theta$

$$\Rightarrow \Gamma(1/2) = \int_0^{\infty} e^{-\theta^2} (\theta^2)^{1/2} \cdot 2\theta \, d\theta$$

$$= -\int e^{-\theta^2} (-2\theta) \cdot \theta \, d\theta$$

$$= -\left[(\theta e^{-\theta^2})_0^{\infty} - \int_0^{\infty} (1) e^{-\theta^2} d\theta \right]$$

$$= \int_0^{\infty} e^{-\theta^2} d\theta$$

$$= \sqrt{\pi}/2$$

Now

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\Gamma(n+s)} \left(\frac{x}{2}\right)^{n+2s}$$

$$J_{1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\Gamma(1/2+s)} \left(\frac{x}{2}\right)^{1/2+2s}$$

$$= \frac{(-1)^0}{\Gamma(0)\Gamma(1/2)}\left(\frac{x}{2}\right)^{1/2} + \frac{(-1)^1}{\Gamma(1)\Gamma(3/2)}\left(\frac{x}{2}\right)^{5/2} + \frac{(-1)^2}{\Gamma(2)\Gamma(5/2)}\left(\frac{x}{2}\right)^{9/2}$$

$$+ \frac{(-1)^3}{\Gamma(3)\Gamma(7/2)}\left(\frac{x}{2}\right)^{13/2} + \ldots$$

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$$= \frac{1}{0!\,\sqrt{\pi}/2} \cdot \frac{x^{1/2}}{\sqrt{2}} - \frac{1}{1!\cdot 3/2\,\Gamma(1/2)} \cdot \frac{x^{5/2}}{2^{5/2}} + \frac{1}{2!\cdot 5/2\cdot 3/2\,\Gamma(1/2)} \cdot \frac{x^{9/2}}{2^{9/2}} - \frac{1}{3!\cdot 7/2\cdot 5/2\cdot 3/2\,\Gamma(1/2)} \cdot \frac{x^{13/2}}{2^{13/2}} + \cdots$$

$$= \sqrt{\frac{2}{\pi}}\,\sqrt{x} - \frac{1}{1\cdot 3/2\cdot\sqrt{\pi}/2} \cdot \frac{x^2\sqrt{x}}{2^2\cdot 2^{1/2}} + \frac{1}{2\cdot 5/2\cdot 3/2\cdot\sqrt{\pi}/2} \cdot \frac{x^4\sqrt{x}}{2^4\sqrt{2}} - \frac{1}{6\cdot 7/2\cdot 5/2\cdot 3/2\cdot\sqrt{\pi}/2} \cdot \frac{x^6\sqrt{x}}{2^6\sqrt{2}} + \cdots$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\,\sin x$$

QUESTION:-

Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

SOLUTION:-

Let us define

$$\Gamma(x) = \int_0^\infty e^{-t}\cdot t^x\,dx, \text{ then}$$

$$\Gamma(x) = x\,\Gamma(x-1) \text{ and}$$

$$\Gamma(-1/2) = \int_0^\infty e^{-t}\cdot t^{-1/2}\,dt$$

put $t = \theta^2 \Rightarrow dt = 2\theta\,d\theta$

$$\Gamma(-1/2) = \int_0^\infty e^{-\theta^2}\cdot(\theta^2)^{1/2}\cdot 2\theta\,d\theta = 2\int_0^\infty e^{-\theta^2}d\theta$$

$$= 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

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Now $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\Gamma(n+s)}\left(\frac{x}{2}\right)^{n+2s}$$

$$J_{-1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\Gamma(-1/2+s)}\left(\frac{x}{2}\right)^{-1/2+2s}.$$

$$= \frac{(-1)^0}{\Gamma(0)\Gamma(-1/2)}\left(\frac{x}{2}\right)^{-1/2} + \frac{(-1)^1}{\Gamma(1)\Gamma(1/2)}\left(\frac{x}{2}\right)^{3/2}$$

$$+ \frac{(-1)^2}{\Gamma(2)\Gamma(3/2)}\left(\frac{x}{2}\right)^{7/2} + \frac{(-1)^3}{\Gamma(3)\Gamma(5/2)}\left(\frac{x}{2}\right)^{11/2} + \cdots$$

$$= \frac{1}{(1)(\sqrt{\pi})}\left(\frac{x}{2}\right)^{-\frac{1}{2}} - \frac{1}{\frac{1}{2}\cdot\sqrt{\pi}}\left(\frac{x}{2}\right)^{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}}$$

$$+ \frac{1}{2!\,\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}\left(\frac{x}{2}\right)^{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}} - \frac{1}{3!\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}\left(\frac{x}{2}\right)^{6}\left(\frac{x}{2}\right)^{-\frac{1}{2}} + \cdots$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cos x.$$

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QUESTION:-

Calculate $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$, $J_{-5/2}(x)$, $J_{7/2}(x)$, $J_{-7/2}(x)$

SOLUTION:- $J_{3/2}(x) = ?$

Using formula

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

put $n = 1/2$

$$\Rightarrow \frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x) \longrightarrow *$$

Since $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

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and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

So $*$ becomes

$$\frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x = \sqrt{\frac{2}{\pi x}} \cos x + J_{3/2}(x)$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x \right]$$

$J_{-3/2}(x)$:

Using formula

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

put $n = -1/2$

$$\frac{-1}{x} J_{-1/2}(x) = J_{-3/2}(x) + J_{1/2}(x) \longrightarrow (i)$$

Since $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ & $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

So, equ (1) becomes

$$\frac{-1}{x} \sqrt{\frac{2}{\pi x}} \cos x = J_{-3/2}(x) + \sqrt{\frac{2}{\pi x}} \sin x$$

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$\Rightarrow \quad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\sin x - \frac{\cos x}{x}\right)$

$J_{5/2}(x) = ?$

By Pure recurrance relation.

$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x).$

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put $n = 3/2$

$\frac{3}{x} J_{3/2}(x) = J_{5/2}(x) + J_{1/2}(x) \longrightarrow$ (i)

Since $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

So, equation (i) becomes

$\frac{3}{x} \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right) = J_{5/2}(x) + \sqrt{\frac{2}{\pi x}} \sin x$

$\Rightarrow J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x^2} \sin x - \frac{3}{x} \cos x - \sin x\right]$

$= \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x\right]$

$J_{-5/2}(x)$

By pure recurrance relation.

$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$

put $n = -3/2$

$\Rightarrow \frac{-3}{x} J_{-3/2}(x) = J_{-1/2}(x) + J_{-5/2}(x)$

Since $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\sin x - \frac{\cos x}{x}\right)$

and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$\Rightarrow J_{-5/2}(x) = \frac{-3}{x}\sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right) - \sqrt{\frac{2}{\pi x}} \cos x$

$= \sqrt{\frac{2}{\pi x}}\left(\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right)$

$J_{7/2}(x) = ?$

By pure recurrance relation:

$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$

put $n = 5/2$

$\frac{5}{x} J_{5/2}(x) = J_{7/2}(x) + J_{3/2}(x)$ ——— (i)

Since $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$

and $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$.

Therefore (i) becomes

$\frac{5}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right] = J_{7/2}(x) + \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

$J_{7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{15-5x^2}{x^3}\sin x - \frac{15\cos x}{x^2} - \frac{\sin x}{x} + \cos x\right]$

$= \sqrt{\frac{2}{\pi x}}\left[\frac{15-5x^2-x^2}{x^3}\sin x + \frac{x^2\cos x - 15\cos x}{x^2}\right]$

$J_{7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{15-6x^2}{x^3}\sin x + \frac{x^2-15}{x^2}\cos x\right]$

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$J_{-7/2}(x) = ?$

By pure recurrance relation:

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

put $n = -5/2$

$$\Rightarrow \frac{-5}{x} J_{-5/2}(x) = J_{-3/2}(x) + J_{-7/2}(x) \quad \text{--- (i)}$$

Since $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3}{x}\sin x - \frac{3-x^2}{x^2}\cos x\right)$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)$$

Therefore (i) becomes

$$-\frac{5}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{3}{x}\sin x - \frac{3-x^2}{x^2}\cos x\right) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right) + J_{-7/2}(x)$$

$$J_{-7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{-15}{x^2}\sin x + \frac{15-5x^2}{x^3}\cos x + \sin x + \frac{\cos x}{x}\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{-15+x^2}{x^2}\sin x + \frac{15-5x^2+x^2}{x^3}\cos x\right]$$

$$J_{-7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{x^2-15}{x^2}\sin x + \frac{15-4x^2}{x^3}\cos x\right]$$

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LEGENDER'S DIFFERENTIAL EQU.

The second order linear differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is called second order LEGENDER differential equation of degree n.

SERIES SOLUTION OF LEGENDER DIFFERENTIAL EQUATION:

Legender differential equation is given by

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \longrightarrow (1)$$

let $y = \sum_{r=0}^{\infty} a_r x^{m+r}$, then

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$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

put in equation (i)

$$(1-x^2)\left[\sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}\right] - 2x\left[\sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}\right] + n(n+1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} - \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r}$$

$$-2\sum_{r=0}^{\infty} a_r (m+r) x^{m+r} + n(n+1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

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$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r \left[n(n+1) - 2(m+r) - (m+r)(m+r-1) \right] x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r \left[n(n+1) - (m+r)(2+m+r-1) \right] x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r \left[n(n+1) - (m+r)(m+r+1) \right] x^{m+r} = 0$$

equating the coefficients of like powers.

coefficient of x^{m-2}: $a_0 m(m-1) = 0 \longrightarrow$ (i)

" " x^{m-1}: $a_1 (m+1) m = 0 \longrightarrow$ (ii)

" " x^{m}: $a_2 (m+2)(m+1) + a_0 [n(n+1) - m(m+1)] = 0 \longrightarrow$ (iii)

" " x^{m+1}: $a_3 (m+3)(m+2) + a_0 [n(n+1) - (m+1)(m+2)] = 0 \longrightarrow$ (iv)

⋮

coefficient of x^{m+r-1}: $a_r (m+r)(m+r-1) x^{m+r-2} + a_{r-2} [n(n+1) - (m+r-2)(m+r-1)] = 0 \longrightarrow$ (v)

from (i)

$a_0 m(m-1) = 0$ As $a_0 \neq 0$

So, $m(m-1) = 0 \Rightarrow m = 0$, $m = 1$

when $m = 0$

from (ii)

~~$a_1 (m+1) m = 0 \Rightarrow a_1 (0+1) \cdot 0 = 0$~~

~~$\Rightarrow a_1 = 0$~~

from (v)

$$a_r = \frac{-n(n+1) - (m+r-2)(m+r-1)}{(m+r)(m+r-1)} a_{r-2}$$

$$a_r = \frac{-n(n+1) - (r-2)(r-1)}{r(r-1)} a_{r-2} \quad ; m = 0$$

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$$a_r = \frac{-(n-r+2)(n+r-1)}{r(r-1)} a_{r-2} \longrightarrow (2)$$

from (2)

put $r=2$

$$a_2 = \frac{-n(n+1)}{2(2-1)} a_0 = \frac{-n(n+1)}{2\cdot 1} a_0$$

Put $r=4$

$$a_4 = \frac{(n-2)(n+3)}{4(3)} a_2 = \frac{(-1)^2 \cdot n(n+1)(n-2)(n+3)}{4\cdot 3\cdot 2\cdot 1} a_0$$

Put $r=6$

$$a_6 = \frac{-(n-4)(n+5)}{6(5)} a_4 = \frac{(-1)^3 n(n+1)(n-2)(n+3)(n-4)(n+5)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} a_0$$

⋮

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Put $r=3$

$$a_3 = \frac{-(n-1)(n+2)}{3\cdot 2} a_1$$

Put $r=5$

$$a_5 = \frac{-(n-3)(n+4)}{5\cdot 4} a_3 = \frac{(-1)^2 (n-1)(n+2)(n-3)(n+4)}{5\cdot 4\cdot 3\cdot 2\cdot 1} a_1$$

put $r=7$

$$a_7 = \frac{(-1)^3 (n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} a_1$$

⋮

Now: $y = \sum a_r x^{m+r} = \sum a_r x^r$, $m=0$

$$\Rightarrow y = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right] +$$

$$a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 + \cdots\right] \longrightarrow (*)$$

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It can be very easily seen that each of the above series converges in $]-1,1[$ by ratio test.
Now by taking $m=1$, we get nothing new but second series of equ $(*)$
Since a_0 and a_1 are arbitrary, therefore this is the general solution of Legender's differential equ. We further note that first series reduces to a polynomial when n is even and second series reduces to a polynomial when n is odd.
Now, If we give such numerical values to a_0 and a_1 so that the polynomial becomes equal to unity when x is unity, then we obtain a system of polynomials

$P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$
$P_3(x)=\frac{1}{2}[5x^3-3x]$, $P_5(x)=\frac{1}{8}[63x^5-70x^3+15x]$
$P_6(x)=\frac{1}{16}[231x^6-315x^4+105x^2-5]$

These polynomials are called Legender's polynomial and each satisfies Legender differential equ, such that the polynomial has the degree equals to the degree of L.D.Eq.
The general form of $P_n(x)$ is given by:

$$P_n(x)=\sum_{r=0}^{r}\frac{(-1)^r(2n-2r)!}{2^n\,r!\,(n-r)!\,(n-2r)!}x^{n-2r}$$

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where $N = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$

Further note that $P_n(x)$ is even or odd according as n is even or odd.

RODRIGUE'S FORMULA

FOR LEGENDER'S FORMULA:-

Prove that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

PROOF:-

Consider Legender's diff equ

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \longrightarrow (1)$$

Let $v = (x^2-1)^n$

$$\Rightarrow \frac{dv}{dx} = n(x^2-1)^{n-1}\cdot 2x$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nxv$$

$$(1-x^2)\frac{dv}{dx} + 2nxv = 0$$

Again differentiating w.r.t 'x'

$$(1-x^2)\frac{d^2v}{dx^2} + (-2x)\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$$

$$\Rightarrow (1-x^2)\frac{d^2v}{dx^2} + (2n-2)x\frac{dv}{dx} + 2nv = 0 \longrightarrow (2)$$

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Again diff w.r.t 'x' we get

$$(1-x^2)\frac{d^3v}{dx^3} + (-2x)\frac{d^2v}{dx^2} + (2n-2)\frac{dv}{dx} + (2n-2)x\frac{d^2v}{dx^2}$$

$$+ 2n\frac{dv}{dx} = 0$$

$$\Rightarrow (1-x^2)\frac{d^3v}{dx^3} + (2nx-2x-2x)\frac{d^2v}{dx^2} + (2n-2+2n)\frac{dv}{dx} = 0$$

$$(1-x^2)\frac{d^3v}{dx^3} + 2x(n-2)\frac{d^2v}{dx^2} + 2(2n-1)\frac{dv}{dx} = 0$$

$$(1-x^2)\frac{d^2(v_1)}{dx^2} + 2x(n-2)\frac{dv_1}{dx} + 2(2n-1)v_1 = 0 \longrightarrow (3)$$

$\because v_1 = dv/dx$

Now (2) can be written as

$$(1-x^2)\frac{d^2v}{dx^2} + 2x(n-0-1)\frac{dv}{dx} + (2n-0)(1+0)v = 0 \rightarrow *$$

(3) can be written as

$$(1-x^2)\frac{d^2v_1}{dx^2} + 2x(n-1-1)\frac{dv_1}{dx} + (2n-1)(1+1)v_1 = 0 \longrightarrow **$$

keeping * and ** in view, if we differentiate equ (2) n-Times.

$$(1-x^2)\frac{d^2v_n}{dx^2} + 2x(n-n-1)\frac{dv_n}{dx} + (2n-n)(n+1)v_n = 0$$

$$(1-x^2)\frac{d^2v_n}{dx^2} - 2x\frac{dv_n}{dx} + n(n+1)v_n = 0$$

This shows that v_n is the solution of Legender's differential equation

But $v_n = \frac{d^nv}{dx^n} = \frac{d^n}{dx^n}(x^2-1)^n$

But solution of Legender's diff equ

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is $P_n(x)$ and is unique, so, then for some c

$$P_n(x) = c \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

$$c \cdot \frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

Equating coefficient of x^n on both side

$$c(2n)(2n-1)(2n-2)----(n+1) = \frac{(-1)^0 (2n-0)!}{2^n \cdot 0!\,(n-0)!\,(n-0)!}$$

$$\frac{c(2n)(2n-1)(2n-2)---(n+1) \cdot n!}{n!} = \frac{2n!}{2^n \cdot n! \cdot n!}$$

$$\frac{c \cdot 2n!}{n!} = \frac{2n!}{2^n n! \, n!} \Rightarrow c = \frac{1}{2^n n!}$$

$$\Rightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

GENERATING FUNCTION OF $P_n(x)$:-

Prove that the coefficient of z^n in $(1-2xz+z^2)^{-1/2}$ is $P_n(x)$.

PROOF:-

$$(1-2xz+z^2)^{-1/2} = [1+(-2xz+z^2)]^{-1/2}$$

$$= 1 + (-1/2)(-2xz+z^2) + \frac{(-1/2)(-1/2-1)}{2!}(-2xz+z^2)^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}(-2xz+z^2)^3 + ------$$

$$= 1 - \frac{1}{2}(-2xz+z^2) + \frac{3}{8}(4x^2z^2+z^4-4xz^3) - \frac{5}{16}(-8x^3z^3+z^6-6xz^5+12x^2z^4) + -----$$

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$= 1 + (x)(z) + [\frac{1}{2}(3x^2-1)]z^2 +$
$[\frac{1}{2}(5x^3-3x)]z^3 + [\frac{1}{8}(35x^4-30x^2+3)]z^4 + \dots$

$[1-2xz+z^2]^{-1/2} = P_0(x)\cdot z^0 + P_1(x)z^1 + P_2(x)z^2 + P_3(x)z^3 + \dots$
$\dots + P_n(x)z^n + \dots$

Hence coefficient of z^n in the expression of $(1-2xz+z^2)^{-1/2}$ is $P_n(x)$.

THEOREM:-

Prove that the Legender's Polynomial are orthogonal.

$$\int_{-1}^{1} P_n(x) \times P_m(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

PROOF:- Let us consider legender's differential equ.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow (1)$$

As, $P_m(x)$ and $P_n(x)$ are the solution of (1) so,

$$(1-x^2)\frac{d^2 P_m(x)}{dx^2} - 2x\frac{d P_m(x)}{dx} + m(m+1)P_m(x) = 0 \rightarrow (2)$$

$$(1-x^2)\frac{d^2 P_n(x)}{dx^2} - 2x\frac{d P_n(x)}{dx} + m(m+1)P_n(x) = 0 \rightarrow (3)$$

multiplying equ (2) by $P_n(x)$ and then subtracting from (3). First multiplying (3) by $P_m(x)$ we get,

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$(1-x^2)[P_n(x)P_m''(x) - P_m(x)P_n''(x)] - 2x[P_n(x)P_m'(x) - P_m(x)P_n'(x)] + [m(m+1) - n(n+1)]P_m(x)P_n(x) = 0$

$\Rightarrow [n(n+1) - m(m+1)]P_m(x)P_n(x) = [(1-x^2)P_n(x)P_m''(x) - 2xP_n(x)P_m'(x)] - [(1-x^2)P_m(x)P_n''(x) - 2xP_m(x)P_n'(x)]$

$= P_n(x)\dfrac{d}{dx}[(1-x^2)P_m'(x)] - P_m(x)\dfrac{d}{dx}[(1-x^2)P_n'(x)]$

$\Rightarrow [n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)P_n(x)\,dx = \int_{-1}^{1}\left\{P_n(x)\dfrac{d}{dx}(1-x^2)P_m'(x) - P_m(x)\dfrac{d}{dx}(1-x^2)P_n'(x)\right\}dx \rightarrow (4)$

Now,

$\int_{-1}^{1} P_n(x)\dfrac{d}{dx}\{(1-x^2)P_m'(x)\}dx = P_n(x)(1-x^2)P_m'(x)\Big|_{-1}^{1} - \int_{-1}^{1} P_n'(x)(1-x^2)P_m'(x)\,dx$

$= 0 - \int_{-1}^{1} P_n'(x)(1-x^2)P_m'(x)\,dx$

$\Rightarrow \int_{-1}^{1} P_n(x)\dfrac{d}{dx}\{(1-x^2)P_m'(x)\}dx = \int_{-1}^{1} P_n'(x)(x^2-1)P_m'(x)\,dx$

Similarly

$\int_{-1}^{1} P_m(x)\dfrac{d}{dx}\{(1-x^2)P_n'(x)\}dx = \int_{-1}^{1}(x^2-1)P_m'(x)P_n'(x)\,dx.$

Hence by (4)

$[n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$

if $m \neq n$, then

$n(n+1) - m(m+1) \neq 0$

$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ for $m \neq n$

Now, we see that what will happen if $m = n$.

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We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)\, t^m.$$

and $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

$$\Rightarrow (1-2xt-t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x) P_n(x)\, t^{m+n}$$

Integrating both sides w.r.t x from -1 to $+1$, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x) P_n(x)\, t^{m+n}\,dx = \int_{-1}^{1} \frac{1}{1-2xt+t^2}\,dx$$

$$= \frac{-1}{2t}\int_{-1}^{1}\frac{-2t}{1-2xt+t^2}\,dx$$

$$= \frac{-1}{2t}\ln\left[1-2xt+t^2\right]_{-1}^{+1}$$

$$= \frac{-1}{2t}\left[\ln(1-2t+t^2)-\ln(1+2t+t^2)\right]$$

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$$= \frac{-1}{2t}\left[\ln(1-t)^2 - \ln(1+t)^2\right]$$

$$= \frac{-1}{t}\left[\ln(1-t) - \ln(1+t)\right]$$

$$= \frac{-1}{t}\left[\left\{-t-\frac{t^2}{2}-\frac{t^3}{3}-\dots\right\}-\left\{t-\frac{t^2}{2}+\frac{t^3}{3}-\dots\right\}\right]$$

$$= \frac{-1}{t}\left[-2t-\frac{2t^3}{3}-\frac{2t^5}{5}-\dots\right]$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)\,t^{m+n}\,dx = 2+\frac{2t^2}{3}+\frac{2t^4}{5}+\dots$$

If $m=n$

$$\sum_{n=0}^{\infty}\int_{-1}^{1}[P_n(x)]^2\, t^{2n}\,dx = 2+\frac{2t^2}{3}+\frac{2t^4}{5}+\dots$$

$$= \frac{2t^{2(0)}}{2(0)+1}+\frac{2t^{2(1)}}{2(1)+1}+\frac{2t^{2(2)}}{2(2)+1}+\dots$$

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Equating coefficient of t^{2n} from both sides

$$\Rightarrow \int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$$

$$\Rightarrow \int_{-1}^{1} P_n(x) P_m(x) \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

RECURRANCE RELATION FOR $P_n(x)$:-

Prove that

1. $P_{n-1}(x) = P_n'(x) - 2x P_{n-1}'(x) + P_{n-2}'(x)$
2. $n P_{n+1}(x) = (2n+1) \cdot x P_n(x) + (n-1) P_{n-1}(x)$

1) $P_{n-1}(x) = P_n'(x) - 2x P_{n-1}'(x) + P_{n-2}'(x)$

PROOF

Consider the generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Differentiating both sides w.r.t 'x', then we have:

$$-\frac{1}{2}(1-2xt+t^2)^{-1/2-1}(-2t) = \sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-1/2} \cdot (1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} P_n'(x) t^n.$$

$$\Rightarrow t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \cdot \sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow t \cdot \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} P_n'(x) t^n - 2xt \sum_{n=0}^{\infty} P_n'(x) t^n + t^2 \sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} P_n'(x) t^n - 2x \sum_{n=0}^{\infty} P_n'(x) t^{n+1} + \sum_{n=0}^{\infty} P_n'(x) t^{n+2}$$

Now, equating the coefficients of t^n

Then we have

$P_{n-1}(x) = P_n'(x) - 2x\, P_{n-1}(x) + P_{n-2}'(x)$

2) $nP_{n+1}(x) = (2n+1)\,xP_n(x) + (n-1)P_{n-1}(x)$

PROOF

Consider the generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

Now, differentiating both sides w.r.t 't' then

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{1}{2}-1}(-2x+2t) = \sum_{n=0}^{\infty} P_n(x)\,nt^{n-1}$$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(1-2xt+t^2)^{-1}(x-t) = \sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(x-t) = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\,t^n(x-t) = \sum_{n=0}^{\infty} P_n(x).nt^{n-1} - 2xt\sum_{n=0}^{\infty} P_n(x).nt^{n-1} + t^2\sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^n x - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=0}^{\infty} P_n(x).nt^{n-1} - 2x\sum_{n=0}^{\infty} P_n(x).nt^n + \sum_{n=0}^{\infty} P_n(x)\,n\,t^{n+1}$$

Now,

Equating the coefficient of t^n

$xP_n(x) - P_{n-1}(x) = n\,P_{n+1}(x) - 2nx\,P_n(x) + P_{n-1}(x).n$

$P_n(x)\,(2n+1).x = n\,P_{n+1}(x) + P_{n-1}(x)\,(n+1)$

$nP_{n+1}(x) = P_n(x)\,(2n+1).x - P_{n-1}(x)\,(n+1)$

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13[illegible]7

Power Series

Method To Find The Solution of LDEq.

LDEq. is given by

$$(1-x^2)\,y''(x) - 2x\,y'(x) + n(n+1)\,y = 0 \rightarrow (i)$$

Let $y = \sum_{m=0}^{\infty} C_m x^m$

$$y' = \sum_{m=0}^{\infty} C_m\, m\, x^{m-1}$$

$$y'' = \sum_{m=0}^{\infty} C_m\, m(m-1)\, x^{m-2}$$

Putting in (i) and simplyfying

$$\sum_{m=2}^{\infty} C_m\, m(1-m)\, x^{m-2} - \sum_{m=0}^{\infty} C_m\, m(m-1)\, x^m - 2\sum_{m=0}^{\infty} C_m\, m\, x^m + n(n+1)\sum_{m=0}^{\infty} C_m x^m = 0$$

Now equating coefficients of x^m

$$C_{m+2}(m+2)(m+1) - C_m\, m(m-1) - 2m\,C_m + n(n+1)\,C_m = 0$$

$$C_{m+2}(m+1)(m+2) - C_m[(m-n)(m+n+1)] = 0$$

$\Rightarrow$

$$C_{m+2} = \frac{(m-n)(m+n+1)}{(m+1)(m+2)}\,C_m \rightarrow (*)$$

$m=0$

$$\Rightarrow C_2 = \frac{(-n)(n+1)}{(1)(2)}\,C_0 = \frac{(-1)^1 n(n+1)}{2!}\,C_0$$

$m=1$

$$\Rightarrow C_3 = \frac{(1-n)(n+2)}{(2)(3)}\,C_1 = \frac{(-1)^1 (n-1)(n+2)}{3!}\,C_1$$

$m=2$

$$\Rightarrow C_4 = \frac{(2-n)(n+3)}{(3)(4)}\,C_2 = \frac{(-1)^2 n(n-1)(n+2)(n+3)}{4!}\,C_0$$

$m=3$

$$C_5 = \frac{(3-n)(n+4)}{(4)(5)}\,C_3 = \frac{(-1)^2 (n-1)(n+2)(n-3)(n+4)}{5!}\,C_1$$

$$y = \sum_{m=0}^{\infty} C_m x^m$$

$$= C_0\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right\} + C_1\left\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 + \cdots\right\}$$

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$\Rightarrow y(x) = C_0 y_1(x) + C_1 y_2(x)$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots$$

&

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} + \cdots$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent therefore it gives two solutions of Legendre's Differential Equation.

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139

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CONVERGENCE OF SERIES

SOLUTION OF LEGENDER'S DIFFERENTIAL EQUATION:

Let us denote the mth term of the series solution, then

$$\left|\frac{U_{m+2}}{U_m}\right| = \left|\frac{C_{m+2}\cdot x^{m+2}}{C_m\cdot x^m}\right|$$

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But $\frac{C_{m+2}}{C_m} = \frac{(m-n)(m+n+1)}{(m+1)(m+2)}$

$$\Rightarrow \left|\frac{U_{m+2}}{U_m}\right| = \left|\frac{C_{m+2}}{C_m}\right| |x^2|$$

$$= \left|\frac{(m-n)(m+n+1)}{(m+1)(m+2}\right| x^2$$

$$\lim_{m\to\infty}\left|\frac{U_{m+2}}{U_m}\right| = \lim_{m\to\infty}\left|\frac{(m-n)(m+n+1)}{(m+1)(m+2)}\right| x^2$$

$$= \lim_{m\to\infty}\left|\frac{(1-n/m)(1+\frac{n+1}{m})}{(1+1/m)(1+2/m)}\right| x^2$$

$$= \left|\frac{(1-0)(1+0)}{(1+0)(1+0)}\right| x^2 = x^2$$

Series solution will converge if $x^2 < 1$

$\Rightarrow -1 < x < 1$

QUESTION:-

Show that $P_n(x) = \sum_{k=0}^{n} [(\binom{n}{k})]^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k$

SOLUTION:- By Roderigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{1}{2^n . n!} \frac{d^n}{dx^n} \left[(x-1)^n (x+1)^n \right]$$

Now by using Leibniz's formula $D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k f \, D^{n-k} g$, we get

$$P_n(x) = \frac{1}{2^n . n!} \sum_{k=0}^{n} \binom{n}{k} D^k (x-1)^n \, D^{n-k} (x+1)^n$$

$$= \frac{1}{2^n . n!} \sum_{k=0}^{n} \binom{n}{k} \left\{ n(n-1)(n-2) \ldots\ldots (n-k+1)(x-1)^{n-k} \right\} . \left\{ n(n-1)(n-2) \ldots\ldots (k+1)(x+1)^k \right\}$$

$$= \frac{1}{2^n . n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} (x-1)^{n-k} . \frac{n! \, (x+1)^k}{k!}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k! (n-k)!} (x-1)^{n-k} (x+1)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} \frac{(x-1)^{n-k}}{2^{n-k}} \frac{(x+1)^k}{2^k}$$

$$= \sum_{k=0}^{n} \left[\binom{n}{k} \right]^2 \left(\frac{x-1}{2} \right)^{n-k} \left(\frac{x+1}{2} \right)^k$$

Question:-

If $f(x) = \sum_{m=0}^{\infty} b_m P_m(x)$, then show that

$$b_m = \frac{2^{m+1}}{2} \int_{-1}^{1} P_m(x) \, dx.$$

Solution:-

Given $f(x) = \sum_{m=0}^{\infty} b_m P_m(x)$

$$f(x) P_n(x) = \sum_{m=0}^{\infty} b_m P_m(x) P_n(x)$$

$$\int_{-1}^{1} f(x) P_n(x) = \sum_{m=0}^{\infty} b_m \int_{-1}^{1} P_m(x) P_n(x) \, dx$$

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$$= b_n \int_{-1}^{1} P_n(x) P_n(x)\, dx$$

$\because P_m(x)$ and $P_n(x)$ are orthogonal

$$= b_n \frac{2}{2n+1}$$

$$\Rightarrow b_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Replacing n by m.

$$\Rightarrow b_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

THEOREM:-

Prove that if n is +ve integer then

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\, dx = \frac{2t^n}{2n+1}$$

and hence using the Roderigue's formula show that

$$\int_{-1}^{1} (1-x^2)^n (1-2xt+t^2)^{-1/2-n}\, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

PROOF:-

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)\, t^m$$

Multiplying both sides by $P_n(x)$ and then integrating from -1 to 1

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2} dx = \sum_{m=0}^{\infty} t^m \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

$$= t^n \int_{-1}^{1} P_n(x) P_n(x)\, dx$$

$\because P_m(x)$ and $P_n(x)$ are orthogonal

$$= t^n \frac{2}{2n+1}$$

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Now by Roderigue's formula

$$\frac{2t^n}{2n+1} = \int_{-1}^{1} \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n (1-2xt+t^2)^{-1/2} dx$$

$$\Rightarrow \frac{2t^n}{2n+1} = \frac{1}{2^n . n!} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n (1-2xt+t^2)^{-1/2} dx$$

$$= \frac{1}{2^n n!}\left[(1-2xt+t^2)^{-1/2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n - \int_{-1}^{1} -\tfrac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right]$$

$$= \frac{1}{2^n n!}\left\{\left[(1-2xt+t^2)^{-1/2} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1}(-\tfrac{1}{2})(1-2xt+t^2)^{-3/2}(-2t) . \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right\}$$

$$= \frac{1}{2^n n!} (-1)^1 . t^1 . 1 \int_{-1}^{1} (1-2xt+t^2)^{-1/2-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

Continuing upto n times integration, we get

$$\frac{2t^n}{2n+1} = \frac{1}{2^n n!} (-1)^n t^n (1)(3)(5)\cdots(2n-1)\int_{-1}^{1}(1-2xt+t^2)^{-1/2-n}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!} (-1)^{2n} t^n \frac{(1)(3)(5)\cdots(2n-1)(2)(4)\cdots 2n}{2.4.6\ldots 2n} \int_{-1}^{1}(1-2xt+t^2)^{-1/2-n}(1-x^2)^n dx$$

$$= \frac{t^n}{2^n n! . 2^n} \frac{1.2.3\ldots(2n-1)(2n)}{1.2.3\ldots n} \int_{-1}^{1}(1-2xt+t^2)^{-1/2-n}(1-x^2)^n dx$$

$$= \frac{t^n (2n)!}{2^{2n} n! n!} \int_{-1}^{1} (1-2xt+t^2)^{-1/2-n}(1-x^2)^n dx$$

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$$\Rightarrow \int_{-1}^{1} (1-2xt+t^2)^{-\frac{1}{2}-n}(1-x^2)^n\, dx = \frac{2t^n}{2n+1}\cdot\frac{2^{2n}(n!)^2}{t^n(2n)!}$$

$$= \frac{2^{2n+1}.(n!)^2}{(2n+1)!}$$

QUESTION:-

Show that if m and n are +ve integers then $\int_{-1}^{1}(1+x)^{m+n} P_n(x)\, dx = \frac{2^{m+n+1}[(m+n)!]^2}{m!\ (m+2n+1)!}$

SOLUTION:-

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By Roderigue's formula

$$P_n(x) = \frac{1}{2^n . n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Now multiplying both sides by $(1+x)^{m+n}$ and then integrating w.r.t 'x'
Then,

$$\int_{-1}^{1}(1+x)^{m+n} P_n(x)\, dx = \int_{-1}^{1}(1+x)^{m+n}\cdot\frac{1}{2^n . n!}\frac{d^n}{dx^n}(x^2-1)^n dx \longrightarrow *$$

$$= \frac{1}{2^n.n!}\left[\left|(1+x)^{m+n}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right|_{-1}^{1} - \int_{-1}^{1}(m+n)(1+x)^{m+n-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right]$$

$$= \frac{1}{2^n.n!}\left[0-0-\int_{-1}^{1}(m+n)(1+x)^{m+n-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right]$$

$$\Rightarrow \int_{-1}^{1}(1+x)^{m+n}P_n(x)\,dx = \frac{-1\cdot(m+n)}{2^n.n!}\int_{-1}^{1}(1+x)^{m+n-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

Now integrating R.H.S w.r.t 'x' upto $n-1$ times
Then

$$\int_{-1}^{1}(1+x)^{m+n}P_n(x)dx = \frac{(-1)^n(m+n)(m+n-1)(m+n-2)\text{-----}(m+1)}{2^n.n!}\int_{-1}^{1}(1+x)^m.(x^2-1)^n dx$$

145

$$\int_{-1}^{1}(1+x)^{m+n}P_n(x)\,dx = \frac{(-1)^n(m+n)(m+n-1)\cdots(m+1)}{2^n.\,n!}\int_{-1}^{1}(1+x)^m(x+1)^n(x-1)^n dx \quad \rightarrow (1)$$

Consider $\int_{-1}^{1}(1+x)^{m+n}(x-1)^n\,dx$

$$= \left|(x-1)^n\frac{(1+x)^{m+n+1}}{m+n+1}\right|_{-1}^{1} - \int_{-1}^{1}\frac{n(x-1)^{n-1}(1+x)^{m+n+1}}{m+n+1}dx$$

$$= [0-0] - \int_{-1}^{1}\frac{n(x-1)^{n-1}(x+1)^{m+n+1}}{m+n+1}dx$$

$$= \frac{(-1).n}{m+n+1}\int_{-1}^{1}(x-1)^{n-1}(x+1)^{m+n+1}\,dx.$$

Now, integrating n times.

$$\Rightarrow \int_{-1}^{1}(1+x)^{m+n}(x-1)^n dx = \frac{(-1)^n.n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\int_{-1}^{1}(x-1)^0(x+1)^{m+2n}dx$$

$$= \frac{(-1)^n n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\int_{-1}^{1}(x+1)^{m+2n}dx$$

$$= \frac{(-1)^n.n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\left|\frac{(x+1)^{m+2n+1}}{m+2n+1}\right|_{-1}^{1}$$

$$= \frac{(-1)^n.n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\left[\frac{2^{m+2n+1}}{m+2n+1}-0\right]$$

$$= \frac{(-1)^n.n!\,.\,2^{m+2n+1}}{(m+n+1)(m+n+2)\cdots(m+2n)(m+2n+1)}$$

Now equ (1) becomes

$$\int_{-1}^{1}(1+x)^{m+n}P_n(x)\,dx = \frac{(-1)^n(m+n)(m+n-1)\cdots(m+1).(-1)^n.n!\,2^{m+2n+1}}{2^n.n!(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}.2^{m+n+1}.2^n(m+n)(m+n-1)\cdots(m+1)}{2^n.(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}.2^{m+n+1}(m+n)(m+n-1)\cdots(m+1)}{(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}.2^{m+n+1}(m+n)(m+n-1)\cdots(m+1)}{(m+2n+1)(m+2n)\cdots(m+n+1)}\times\frac{(m+n)!}{(m+n)!}$$

$$\times\frac{m(m-1)\cdots3.2.1}{m(m-1)\cdots3.2.1}$$

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$$= \frac{(-1)^{2n}\, 2^{m+n+1} (m+n)(m+n-1)\cdots(m+1)\, m(m-1)\cdots 3.2.1\,(m+n)!}{(m+2n+1)(m+2n)\cdots(m+n+2)(m+n)!\, m(m-1)\cdots 3.2.1}$$

$$= \frac{(-1)^{2n}. 2^{m+n+1}\left[(m+n)(m+n-1)\cdots(m+1)(m)(m-1)\cdots 3.2.1\right](m+n)!}{\left[(m+2n+1)(m+2n)\cdots(m+n+2)(m+n)(m+n-1)\cdots m(m-1)\cdots 3.2.1\right] m(m-1)\cdots 3.2.1}$$

$$= \frac{(-1)^{2n} . 2^{m+n+1} (m+n)!\,(m+n)!}{(m+2n+1)! . m!}$$

$$\Rightarrow \int_{-1}^{1} (1+x)^{m+n} P_n(x)\, dx = \frac{(-1)^{2n}\, 2^{m+n+1} \left[(m+n)!\right]^2}{m!\,(m+2n+1)!}$$

$$\Rightarrow \int_{-1}^{1} (1+x)^{m+n} P_n(x)\, dx = \frac{2^{m+n+1} \left[(m+n)!\right]^2}{m!\,(m+2n+1)!}$$

$\because \quad (-1)^{2n} = 1$

QUESTION:-

If $q = x + \sqrt{x^2-1}$, then show that

$$(1-tq)^{-1/2}(1-t/q)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

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SOLUTION:-

$$L.H.S = (1-tq)^{-1/2}(1-t/q)^{-1/2}$$

$$= (1-tq)^{-1/2}(q-t)^{-1/2}.(1/q)^{-1/2}$$

$$= \sqrt{q}\,\left[(1-tq)(q-t)\right]^{-1/2}$$

$$= \sqrt{q}\,\left[q - t + t^2 q - tq^2\right]^{-1/2}$$

$$= \sqrt{q}\,\left[x + \sqrt{x^2-1} - t + t^2(x+\sqrt{x^2-1}) - t\,(x+\sqrt{x^2-1})^2\right]$$

$$= \sqrt{q}\,[x+\sqrt{x^2-1} - t - t(x^2+(x^2-1)+2x\sqrt{x^2-1}) + t^2(x+\sqrt{x^2-1})]^{-1/2}$$

$$= \sqrt{q}\,[x+\sqrt{x^2-1} - t - 2tx^2 + t - 2xt\sqrt{x^2-1} + t^2x + t^2\sqrt{x^2-1}]^{-1/2}$$

$$= \sqrt{q}\,[x+\sqrt{x^2-1} - 2tx(x+\sqrt{x^2-1}) + t^2(x+\sqrt{x^2-1})]^{-1/2}$$

$$= \sqrt{q}\,[q - 2txq + t^2q]^{-1/2}$$

$$= (q)^{1/2}(q)^{-1/2}(1-2tx+t^2)^{-1/2}$$

$$= (1-2tx+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

$$= R.H.S.$$

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148

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HYPERGEOMETRIC DIFFERENTIAL EQUATION:-

The differential equation

$$z(1-z)\,y''(z) + [\gamma - (\alpha+\beta+1)z]\,y'(z) - \alpha\beta\, y(z) = 0$$

is called hypergeometric differential equation.

HYPERGEOMETRIC FUNCTION:- 2011, 2016

The function denoted and defined by

$$F_{2,1}(\alpha,\beta,\gamma\,;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}$$

is called hypergeometric function, where α, β, γ are real parameters and z is a variable and

$$(\alpha)_n = (\alpha)(\alpha+1)(\alpha+2)\,----\,(\alpha+n-1)$$

and $(\alpha)_n$, $(\beta)_n$ and $(\gamma)_n$ are called christopher symbols. Further $(\alpha)_0 = 1$
Morever, γ is neither zero nor any -ve integer and the suffix 2 and 1 means that two parameters are in the numerator and one parameter in the denuminator.

The number of parameters in the numerator and in the denominator, and further more the number of variables can vary But here we will discuss the case only described above.
The hypergeometric function is also denoted by

$$F\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix}\; z\right]$$

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Series Solution of Hypergeometric Differential Equation:-

Consider Hypergeometric differential eqn.

$$z(1-z)\,y''(z) + [\gamma - (\alpha+\beta+1)z]\,y'(z) - \alpha\beta\, y(z) = 0 \rightarrow (1)$$

let $y = \sum_{m=0}^{\infty} C_m z^{m+n}$

Then (1) becomes

$$\sum_{m=0}^{\infty} C_m[(m+n)(m+n-1) + \gamma(m+n)]\, z^{m+n-1} - \sum_{m=0}^{\infty} C_m[(m+n)(m+n+\alpha+\beta) + \alpha\beta]\, z^{m+n} = 0 \longrightarrow *$$

Now equating coefficients of like powers

coefficients of z^{n-1}: $C_0[n(n-1) + \gamma n] = 0 \longrightarrow$ (i)

" " z^{n}: $C_1[(1+n)n + \gamma(1+n)] - [n(n+\alpha+\beta) + \alpha\beta]C_0 = 0 \longrightarrow$ (ii)

" " z^{n+1}: $C_2[(2+n)(1+n) + \gamma(2+n)] - C_1[(1+n)(1+n+\alpha+\beta) + \alpha\beta] = 0 \longrightarrow$ (iii)

⋮

" " z^{m+n-1}: $C_m[(m+n)(m+n-1) + \gamma(m+n)] - C_{m-1}[(m-1+n)(m-1+n+\alpha+\beta) + \alpha\beta] = 0 \longrightarrow$ (iv)

From (i)

$C_0[n(n-1) + \gamma n] = 0$

As $C_0 \neq 0 \Rightarrow n(n-1) + \gamma n = 0$

$n(n-1+\gamma) = 0$

$\Rightarrow n = 0,\ n-1+\gamma = 0$

$\Rightarrow n = 0,\ n = 1-\gamma$

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when $n=0$

From (iv)

$$C_m = \frac{(m+n-1)(m+n+\alpha+\beta-1)+\alpha\beta}{(m+n)(m+n-1)+\gamma(m+n)} C_{m-1}$$

Since $n=0$, So

$$C_m = \frac{(m-1)(m+\alpha+\beta-1)+\alpha\beta}{m(m-1)+\gamma\cdot(m)} C_{m-1}$$

$$\Rightarrow C_m = \frac{(\alpha+m-1)(\beta+m-1)}{m(\gamma+m-1)} C_{m-1} \longrightarrow (2)$$

Put $m=1$

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$$\Rightarrow C_1 = \frac{(\alpha)(\beta)}{1\cdot\gamma} C_0$$

Put $m=2$, $\Rightarrow C_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} C_1$

$$= \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} \cdot \frac{\alpha\cdot\beta}{\gamma} C_0$$

$$= \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{2\cdot1\;\gamma(\gamma+1)} C_0$$

Put $m=3$

$$\Rightarrow C_3 = \frac{(\alpha+2)(\beta+2)}{3(\gamma+2)} C_2$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{3\cdot2\cdot1\;\gamma(\gamma+1)(\gamma+2)} C_0$$

Continuing in this way

$$C_m = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\beta(\beta+1)\cdots(\beta+m-1)}{m(m-1)\cdots3\cdot2\cdot1\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+(m-1))} C_0$$

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$$= \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} c_0$$

Thus $y(z) = \sum_{m=0}^{\infty} c_m z^{m+n} = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} z^{m+0} c_0$

$\because n=0.$

Choose

$$y(z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} z^m = F_{2,1}[\alpha, \beta, \gamma ; z]$$

which is required solution of Hypergeometric differential equ.

Next, when $n = 1-\gamma$, then let $y(z) = z^{1-\gamma} v(z)$ where $v(z)$ is a solution of ① for $n=0$.

Then $y'(z) = z^{1-\gamma} v'(z) + (1-\gamma) z^{-\gamma} v(z)$

& $y''(z) = z^{1-\gamma} v''(z) + (1-\gamma) z^{-\gamma} v'(z) + (1-\gamma)(-\gamma) z^{-\gamma-1} v(z) + (1-\gamma) z^{-\gamma} v'(z)$.

Putting these values in equ ① and after some simplification we get

$(-\gamma)(1-\gamma) z^{-\gamma}(1-z) v(z) + 2(1-\gamma) z^{1-\gamma}(1-z) v'(z) + z^{2-\gamma} v''(z) + \gamma(1-\gamma) z^{-\gamma} v(z) + \gamma z^{1-\gamma} v'(z) - (\alpha+\beta+1)(1-\gamma) z^{1-\gamma} v(z) - (\alpha+\beta+1) z^{2-\gamma} v'(z) - \alpha\beta z^{1-\gamma} v(z) = 0$

Dividing by $z^{-\gamma}$ and after some simplification we get

$[z^2(1-z)] v''(z) + [2(1-\gamma) z(1-z) + \gamma z - (\alpha+\beta+1) z^2] v'(z) + [(-\gamma)(1-\gamma)(1-z) + \gamma(1-\gamma) - (\alpha+\beta+1)(1-\gamma) z - \alpha\beta z] v(z) = 0$

$\Rightarrow [z^2(1-z)] v''(z) + [2(1-\gamma) z(1-z) + \gamma z - (\alpha+\beta+1) z^2] v'(z) + [\gamma(1-\gamma) z - (\alpha+\beta+1)(1-\gamma) z - \alpha\beta z] v(z) = 0$

Dividing by z.

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$\Rightarrow z(1-z)v''(z) + [2(1-\gamma)(1-z)+\gamma-(\alpha+\beta+1)z]v'(z) + [\gamma(1-\gamma)-(\alpha+\beta+1)(1-\gamma)-\alpha\beta]v(z) = 0$

$\Rightarrow z(1-z)v''(z) + [(2-\gamma)-\{(\alpha-\gamma+1)+(\beta-\gamma+1)+1\}]v'(z) - (\alpha-\gamma+1)(\beta-\gamma+1)v(z) = 0 \longrightarrow (3)$

(3) is now hypergeometric differential equ:

For $n = 0$

$$v(z) = \sum_{m=0}^{\infty} \frac{(\alpha-\gamma+1)_m (\beta-\gamma+1)_m}{(2-\gamma)_m \cdot m!} z^m$$

$$\Rightarrow y(z) = z^{1-\gamma} v(z) = z^{1-\gamma} \cdot \sum_{m=0}^{\infty} \frac{(\alpha-\gamma+1)_m(\beta-\gamma+1)_m}{(2-\gamma)_m \cdot m!} z$$

$$= z^{1-\gamma} F_{2,1}[\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z]$$

which is required solution of hypergeometric differential equ for $n = 1-\gamma$.

CONVERGENCE OF SERIES SOLUTION OF HYPERGEOMETRIC DIFFERENTIAL EQU.

By Ratio test

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(\alpha)_{n+1} \cdot (\beta)_{n+1}}{(\gamma)_{n+1} \cdot (n+1)!} z^{n+1} \cdot \frac{(\gamma)_n\, n!}{(\alpha)_n \cdot (\beta)_n} \cdot \frac{1}{z^n}\right|$$

$$= \left|\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)} z\right|$$

$$= \frac{(\alpha/n+1)(\beta/n+1)}{(\gamma/n+1)(1+1/n)} |z|$$

when $n \to \infty$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \frac{(0+1)(0+1)}{(0+1)(1+0)}|z| = |z|$$

$\Rightarrow$ Series solution converges if $|z| < 1 \Rightarrow -1 < z < 1$

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GAMMA FUNCTION:

$$\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1} \cdot dt$$

and

$$\Gamma(x) = (x-1)\,\Gamma(x-1)$$

and if x is non negative integer, then

$$\Gamma x = (x-1)!$$

BETA FUNCTION:

$$\beta(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1}\, dx$$

Further $\beta(l,m) = \dfrac{\Gamma l\, \Gamma m}{\Gamma l+m}$.

QUESTION:-

Show that $(1-x)^{-\alpha} = \sum_{m=0}^{\infty} \dfrac{(\alpha)_m x^m}{m!}$

SOLUTION:-

$$(1-x)^{-\alpha} = 1 + (-\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}(-x)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}(-x)^3 + \cdots$$

$$= 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \cdots$$

$$= \frac{(\alpha)_0 x^0}{0!} + \frac{(\alpha)_1 x^1}{1!} + \frac{(\alpha)_2 x^2}{2!} + \frac{(\alpha)_3 x^3}{3!} + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m \cdot x^m}{m!}$$

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QUESTION:- $\dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$

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Sol: $L.H.S = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

$= \dfrac{(\alpha+n-1)\,\Gamma(\alpha+n-1)}{\Gamma(\alpha)}$

$= \dfrac{(\alpha+n-1)(\alpha+n-2) \,\text{-----}\, (\alpha+2)(\alpha+1)\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)}$

$= \alpha(\alpha+1)(\alpha+2) \,\text{-----}\, (\alpha+n-2)(\alpha+n-1)$

$= (\alpha)_n$

$L.H.S = R.H.S.$

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INTEGRAL REPRESENTATION OF HYPERGEOMETRIC FUNCTION:

Consider $\beta(\alpha+n, \gamma-\alpha) = \dfrac{\Gamma(\alpha+n)\cdot\Gamma(\gamma-\alpha)}{\Gamma(\alpha+n+\gamma-\alpha)}$

$\because \beta(l,m) = \dfrac{\Gamma l\,\Gamma m}{\Gamma l+m}$

$\Rightarrow \beta(\alpha+n, \gamma-\alpha) = \dfrac{\Gamma(\alpha+n)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma+n)} \cdot \dfrac{\Gamma(\gamma)}{\Gamma(\gamma)} \cdot \dfrac{\Gamma(\alpha)}{\Gamma(\alpha)}$

$= \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \dfrac{\Gamma(\gamma-\alpha)\cdot\Gamma(\alpha)}{\Gamma(\gamma)} \cdot \dfrac{\Gamma(\gamma)}{\Gamma(\gamma+n)}$

$= (\alpha)_n \cdot \dfrac{\Gamma(\gamma-\alpha)\cdot\Gamma(\alpha)}{\Gamma(\gamma)} \cdot \dfrac{1}{(\gamma)_n}$

$\Rightarrow \dfrac{(\alpha)_n}{(\gamma)_n} \dfrac{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)}{\Gamma\gamma} = \beta(\alpha+n, \gamma-\alpha)$

$= \displaystyle\int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}\,dt \longrightarrow (1)$

By def of $\beta(l,m)$

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Now

$$F_{2,1}[\alpha,\beta,\gamma;z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\, n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n z^n}{n!} \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}dt$$

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By equ (1)

$$= \sum_{n=0}^{\infty} \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \int_0^1 t^{\alpha-1} t^n (1-t)^{\gamma-\alpha-1} \frac{(\beta)_n z^n}{n!} dt$$

$$= \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\left(\sum_{n=0}^{\infty} \frac{(\beta)_n (tz)^n}{n!}\right)dt$$

$$\Rightarrow F_{2,1}[\alpha,\beta,\gamma;z] = \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}dt$$

$$\because (1-t)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!}$$

which is required integral representation of Hypergeometric function.

QUESTION:-

Show that $F_{2,1}[\alpha,\beta,\gamma;1] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$

Sol:

By integral representation of Hypergeometric function

$$F_{2,1}[\alpha,\beta,\gamma;z] = \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}dt$$

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$$\Rightarrow F_{2,1}[\alpha,\beta,\gamma;1] = \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-t)^{-\beta}dt$$

$$= \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-t)^{(\gamma-\alpha-\beta)-1}dt$$

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$$= \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \beta(\alpha, \gamma-\alpha-\beta)$$

$$= \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\Gamma\alpha} \cdot \frac{\Gamma\alpha \cdot \Gamma(\gamma-\alpha-\beta)}{\Gamma(\alpha+\gamma-\alpha-\beta)}$$

$$= \frac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

QUESTION:-

Show that

(i) $F(-n, \alpha+n, \gamma; 1) = \dfrac{(\gamma-\alpha-n)_n}{(\gamma)_n}$

(ii) $F(-n, 1-\beta-n, \alpha; 1) = \dfrac{(\alpha+\beta-1)_{2n}}{(\alpha)_n(\alpha+\beta-1)_n}$

(iii) $\dfrac{d}{dz}[F(\alpha,\beta,\gamma;z)] = \dfrac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1; z)$

SOLUTION:-

(i) $F(-n, \alpha+n, \gamma; 1) = \dfrac{(\gamma-\alpha-n)_n}{(\gamma)_n}$

Sol. Since $(\alpha,\beta,\gamma;1) = \dfrac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \longrightarrow (1)$

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Now put $\alpha = -n$, $\beta = \alpha + n$ in (i)

$$\Rightarrow F(-n, \alpha+n, \gamma; 1) = \frac{\Gamma\gamma \, \Gamma(\gamma - (-n) - (\alpha+n))}{\Gamma(\gamma-(-n)) \, \Gamma(\gamma - (\alpha+n))}$$

$$= \frac{\Gamma\gamma \, \Gamma(\gamma + n - \alpha - n)}{\Gamma(\gamma+n) \, \Gamma(\gamma - \alpha - n)}$$

$$= \frac{1}{\frac{\Gamma(\gamma+n)}{\Gamma\gamma}} \cdot \frac{\Gamma((\gamma-\alpha-n)+n)}{\Gamma(\gamma-\alpha-n)}$$

$$= \frac{1}{(\gamma)_n} \cdot (\gamma-\alpha-n)_n$$

$$\Rightarrow F[-n, \alpha+n, \gamma; 1] = \frac{(\gamma-\alpha-n)_n}{(\gamma)_n}$$

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(ii) $F(-n, 1-\beta-n, \alpha; 1) = \dfrac{(\alpha+\beta-1)_{2n}}{(\alpha)_n \, (\alpha+\beta-1)_n}$

Sol.

Since $F[\alpha, \beta, \gamma; 1] = \dfrac{\Gamma\gamma \, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha) \, \Gamma(\gamma-\beta)} \longrightarrow$ (i)

Put $\alpha = -n$, $\beta = 1-\beta-n$, $\gamma = \alpha$ in (i)

$$\Rightarrow F[-n, 1-\beta-n, \alpha; 1] = \frac{\Gamma\alpha \, \Gamma(\alpha - (-n) - (1-\beta-n))}{\Gamma(\alpha-(-n)) \, \Gamma(\alpha - (1-\beta-n))}$$

$$= \frac{\Gamma\alpha}{\Gamma(\alpha+n)} \cdot \frac{\Gamma(\alpha+n-1+\beta+n)}{\Gamma(\alpha-1+\beta+n)}$$

$$= \frac{1}{\frac{\Gamma(\alpha+n)}{\Gamma\alpha}} \cdot \frac{\Gamma(\alpha+\beta-1+2n)}{\Gamma(\alpha+\beta-1)+n} \times \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1)}$$

$$= \frac{1}{(\alpha)_n} \cdot \frac{\Gamma(\alpha+\beta-1+2n)}{\Gamma(\alpha+\beta-1)} \cdot \frac{1}{\frac{\Gamma(\alpha+\beta-1)+n}{\Gamma(\alpha+\beta-1)}}$$

$$\Rightarrow F(-n, 1-\beta-n, \alpha; -1) = \frac{1}{(\alpha)_n} \cdot \frac{(\alpha+\beta-1)_{2n}}{(\alpha+\beta-1)_n}$$

(iii) $\frac{d}{dz}[F(\alpha,\beta,\gamma;z)] = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1; z)$

Sol

Since $F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} z^n$

$$\Rightarrow \frac{d}{dz}[F(\alpha,\beta,\gamma;z)] = \frac{d}{dz}\left[\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} z^n\right]$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} n z^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (n-1)!} z^{n-1}$$

$$\Rightarrow \frac{d}{dz}[F(\alpha,\beta,\gamma;z)] = \frac{(\alpha)_1(\beta)_1}{(\gamma)_1 (1-1)!} z^{1-1} + \frac{(\alpha)_2(\beta)_2}{(\gamma)_2(2-1)!} z^{2-1} + \frac{(\alpha)_3(\beta)_3}{(\gamma)_3(3-1)!} z^{3-1} + \frac{(\alpha)_4(\beta)_4}{(\gamma)_4(4-1)!} z^{4-1} + \cdots$$

$$= \frac{\alpha\beta}{\gamma} + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)} z + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 2!} + \cdots$$

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$$= \frac{\alpha\beta}{\gamma}\left[1 + \frac{(\alpha+1)_1(\beta+1)_1}{(\gamma+1)_1} z + \frac{(\alpha+1)_2(\beta+1)_2}{(\gamma+1)_2 \cdot 2!} z^2 + \cdots\cdots\right]$$

$$= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)_n(\beta+1)_n}{(\gamma+1)_n \cdot n!} z^n$$

$$\Rightarrow \frac{d}{dz}\left[F(\alpha,\beta,\gamma;1)\right] = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1; z)$$

QUESTION:

Show that $(\alpha)_{n-k} = \dfrac{(\alpha)_n}{(-1)^k(1-\alpha-n)_k}$ and

hence deduce the result

$$(1)_{n-k} = \frac{n!}{(-1)^k(-n)_k}$$

Sol.

Since $(\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2) \quad (\alpha+(n-k)-1)$

$$\Rightarrow (\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2)\cdots\cdots(\alpha+n-k-1) \times \frac{(\alpha+n-k)(\alpha+n-k+1)\cdots\cdots(\alpha+n-1)}{(\alpha+n-k)(\alpha+n-k+1)\cdots\cdots(\alpha+n-1)}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\cdots\cdots(\alpha+n-k-1)(\alpha+n-k)(\alpha+n-k+1)\cdots(\alpha+n-1)}{(\alpha+n-k)(\alpha+n-k+1)\cdots\cdots(\alpha+n-1)}$$

$$= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)(2-\alpha-n)\cdots\cdots(k-\alpha-n)}$$

$$= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)(1-\alpha-n+1)\cdots\cdots(1-\alpha-n+(k-1))}$$

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$$= \frac{(\alpha)_n}{(-1)^k (1-\alpha-n)_k}$$

Now put $\alpha = 1$

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$$\Rightarrow (1)_{n-k} = \frac{(1)_n}{(-1)^k (-n)_k}$$

$$= \frac{(1)(1+1)(1+2)\cdots(1+(n-1))}{(-1)^k (-n)_k}$$

$$\Rightarrow (1)_{n-k} = \frac{n!}{(-1)^k (-n)_k}$$

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HYPERGEOMETRIC FORM OF LEGENDER'S POLYNOMIAL:

We know that $\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2}$.

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^n = \left[(1+t^2-2t) - 2xt + 2t\right]^{-1/2}$$

$$= \left[(1-t)^2 - 2t(x-1)\right]^{-1/2}$$

$$= \left[(1-t)^2\right]^{-1/2} \left[1 - \frac{2t(x-1)}{(1-t)^2}\right]^{-1/2}$$

$$= (1-t)^{-1} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left[\frac{2t(x-1)}{(1-t)^2}\right]^k$$

$$\because (1-x)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\alpha_k x^k}{k!} \rightarrow *$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k 2^k t^k (x-1)^k}{k!} (1-t)^{-(2k+1)}$$

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$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1/2)_k 2^k t^k}{k!} (x-1)^k \frac{(2k+1)_n t^n}{n!}$$

Again by *

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1/2)_k 2^k t^{n+k} (x-1)^k}{k!\,.\,n!} \frac{(2k+n)!}{(2k)!} \longrightarrow (1)$$

$$\left(\because (2k+1)_n = \frac{\Gamma(2k+1+n)}{\Gamma(2k+1)} = \frac{\Gamma 2k+n+1}{\Gamma(2k+1)} = \frac{(2k+n)!}{(2k)!}\right)$$

Now
$$\frac{(1/2)_k 2^k}{(2k)!} = \frac{(1/2)(1/2+1)(1/2+2) \cdots\cdots (1/2+k-1)\,.\,2^k}{(2k)(2k-1)(2k-2)\cdots\cdots 4.3.2.1}$$

$$= \frac{(1)(3)(5) \cdots\cdots (2k-1)\cdot \frac{1}{2^k}\cdot 2^k}{(2k)(2k-2)(2k-4)\cdots\cdots 4.2\,(2k-1)(2k-3)(2k-5)\cdots\cdots 5.3.1}$$

$$\Rightarrow \frac{(1/2)_k . 2^k}{(2k)!} = \frac{1}{2^k\, k!}$$

Hence by (1)

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^{n+k} (x-1)^k}{k!\,.\,n!} \frac{(2k+n)!}{2^k . k!}$$

Now Replacing n by n-k on R.H.S

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n . (x-1)^k}{k!\,(n-k)!} \frac{(k+n)!}{2^k . k!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n (x-1)^k}{(k!)^2\, 2^k} \frac{(n+k)!}{(n-k)!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n (x-1)^k}{(k!)^2 . 2^k} . (-1)^k (-n)_k (n+1)_k$$

(by previous result)

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$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k}{(1)_k \cdot k!} \left(\frac{1-x}{2}\right)^k \right] t^n$$

$\because (1)_k = k!$

$$= \sum_{n=0}^{\infty} \left[F_{2,1} \left(-n, n+1, 1 ; \frac{1-x}{2}\right)\right] t^n$$

which is required Hypergeometric form of Legendre's polynomial.

i.e

$$P_n(x) = F_{2,1} \left[-n, n+1, 1 ; \frac{1-x}{2}\right]$$

Now replace x by $-x$ on both sides

$$P_n(-x) = F_{2,1} \left[-n, n+1, 1 ; \frac{1+x}{2}\right]$$

$$(-1)^n P_n(x) = F_{2,1} \left[-n, n+1, 1 ; \frac{1+x}{2}\right]$$

$$\Rightarrow P_n(x) = (-1)^n F_{2,1} \left[-n, n+1, 1 ; \frac{1+x}{2}\right]$$

which is another hypergeometric form.

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